Minimum Energy Encoding for Networked Control Systems

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UCSB
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Outline

- Prior work
- Problem setup
- Necessary & Sufficient main result
- Event-based encoding

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- **Prior work**
- Problem setup
- Necessary & Sufficient main result
- Event-based encoding

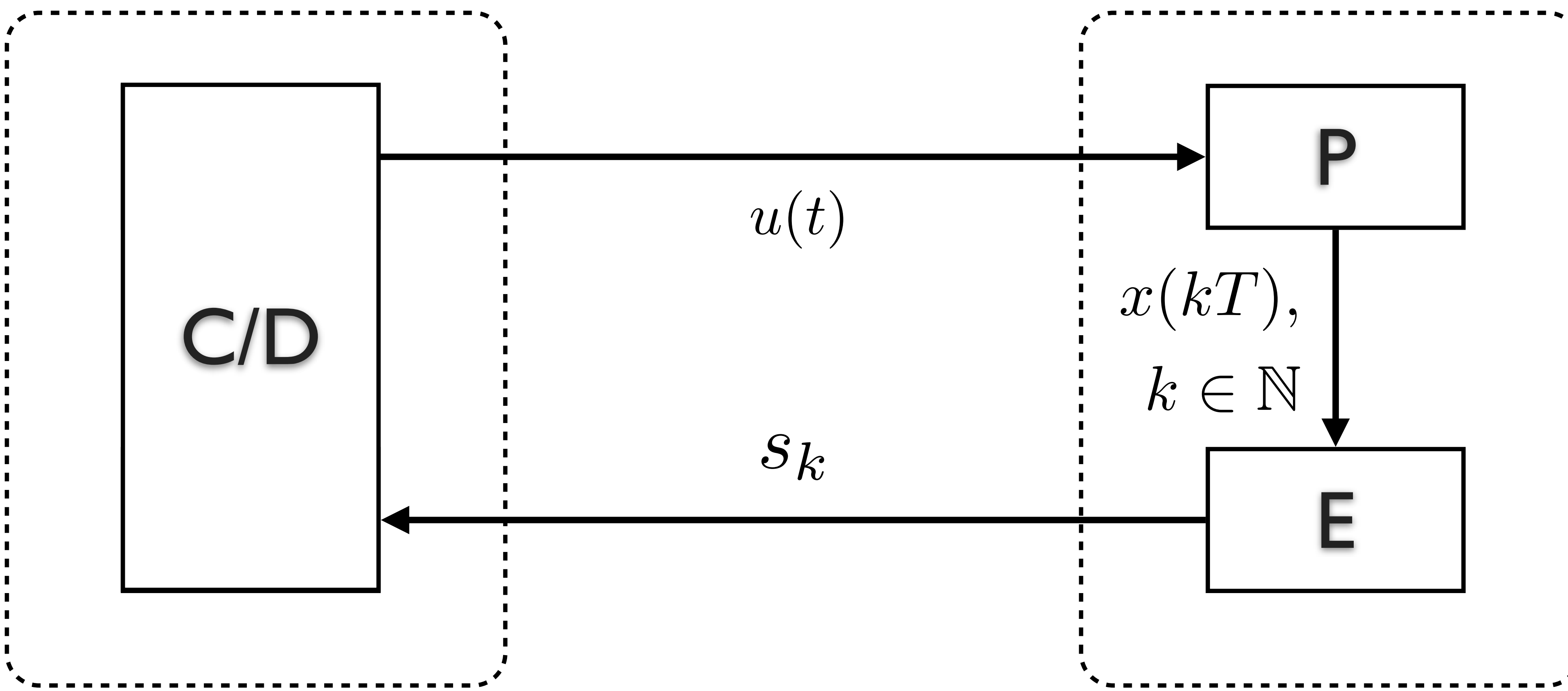
Prior work

- Linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n \text{ bdd}$$

- Encoder/Decoder with alphabet $\mathcal{A}$, sampling period T

$$s_k \in \mathcal{A} := \{0, \ldots, S\}, \quad k \in \mathbb{N}, \qquad r := \frac{\log_2(S+1)}{T}$$



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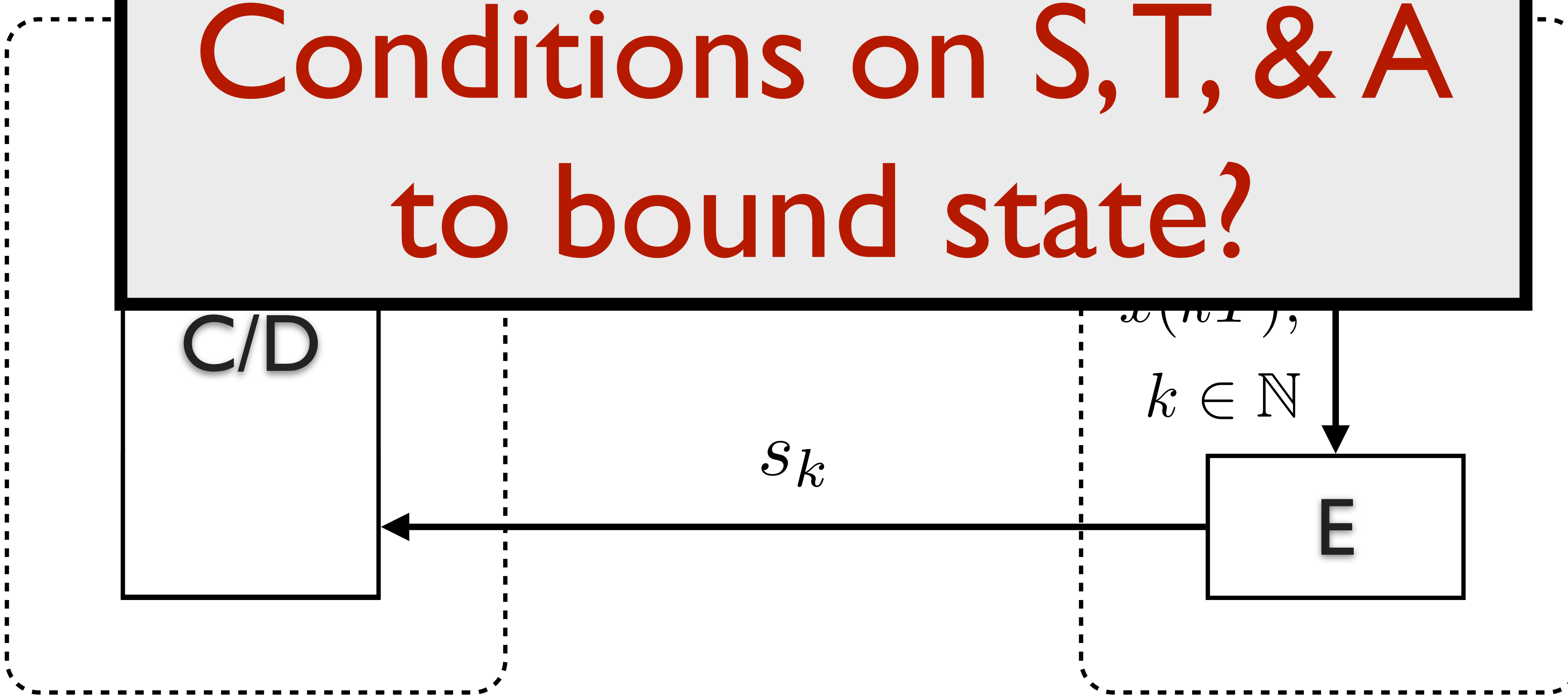


Conditions on S, T, & A to bound state?

Prior work

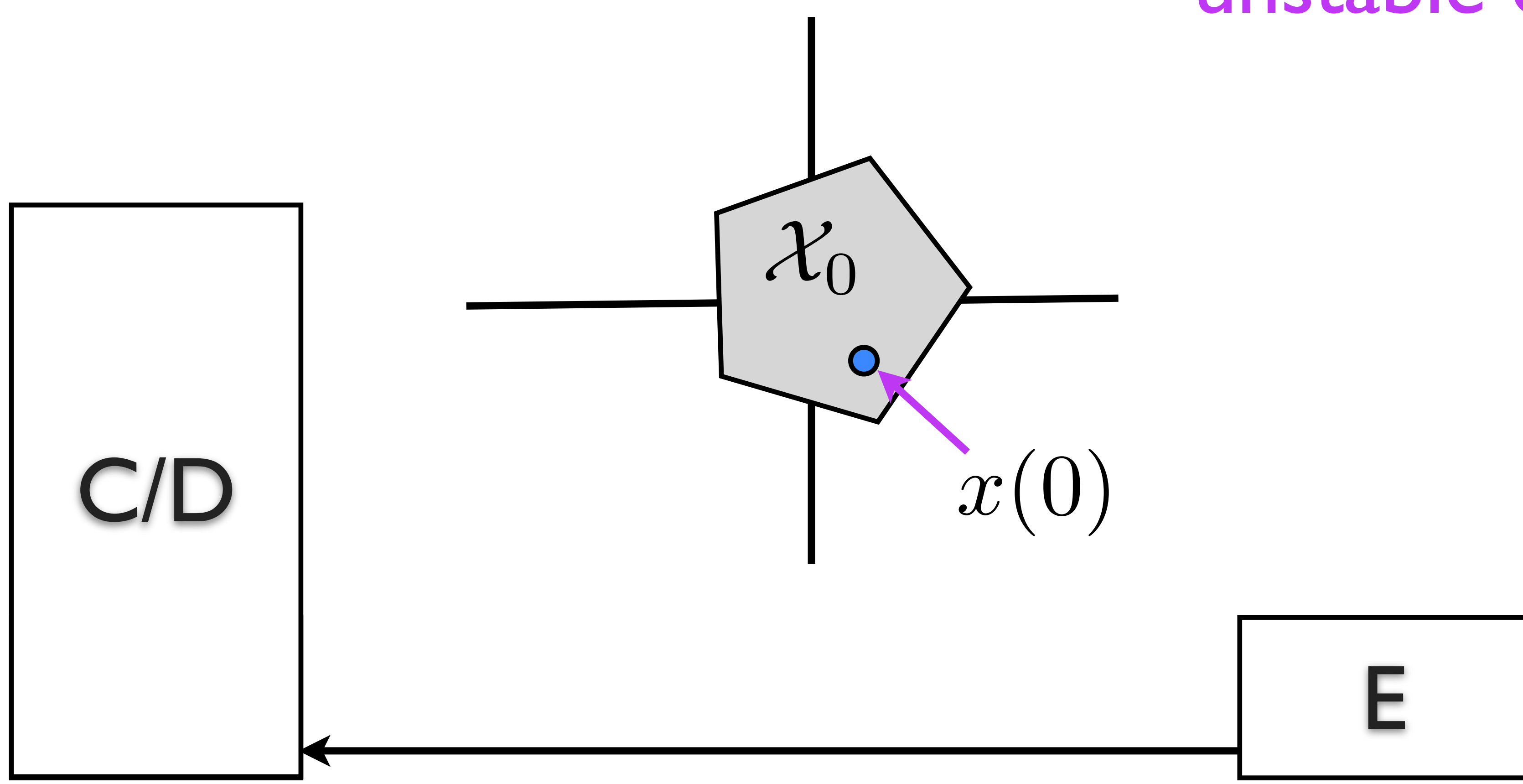
- Channel Capacity with Free Symbols
  - Verdu 1990
- Control under Communication Constraints
  - Brocket/Liberzon 1998
  - Tatikonda/Mitter 2000
  - Nair/Evans 2000
  - Hespanha/Ortega/Vasudevan 2002
  - Li/Baillieul 2005
- Event-based Control
  - Astrom/Bernhardsson 2002
  - Tabuada 2006

Prior work

$$r \ln 2 > \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

bit-rate

unstable evals



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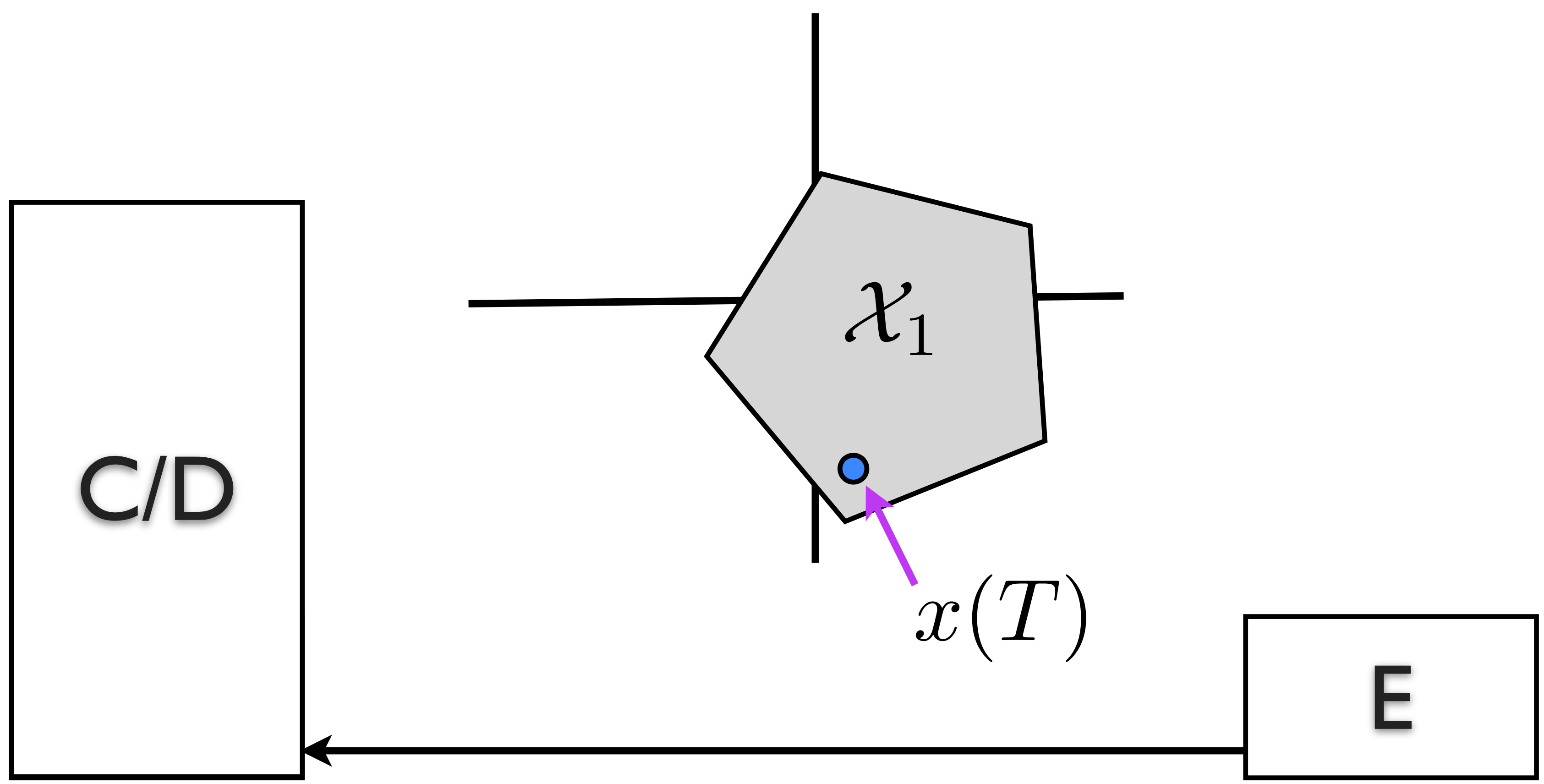

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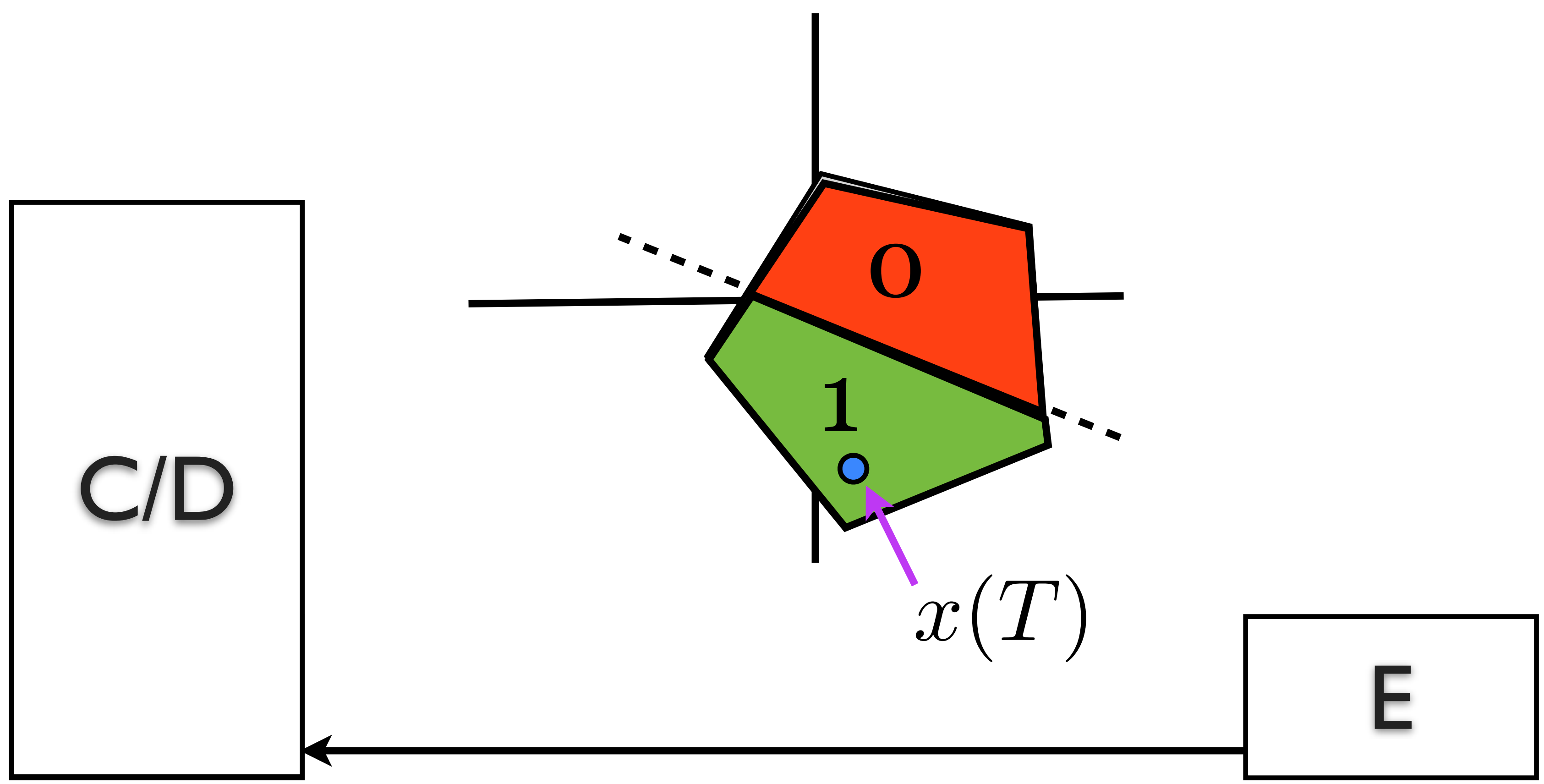

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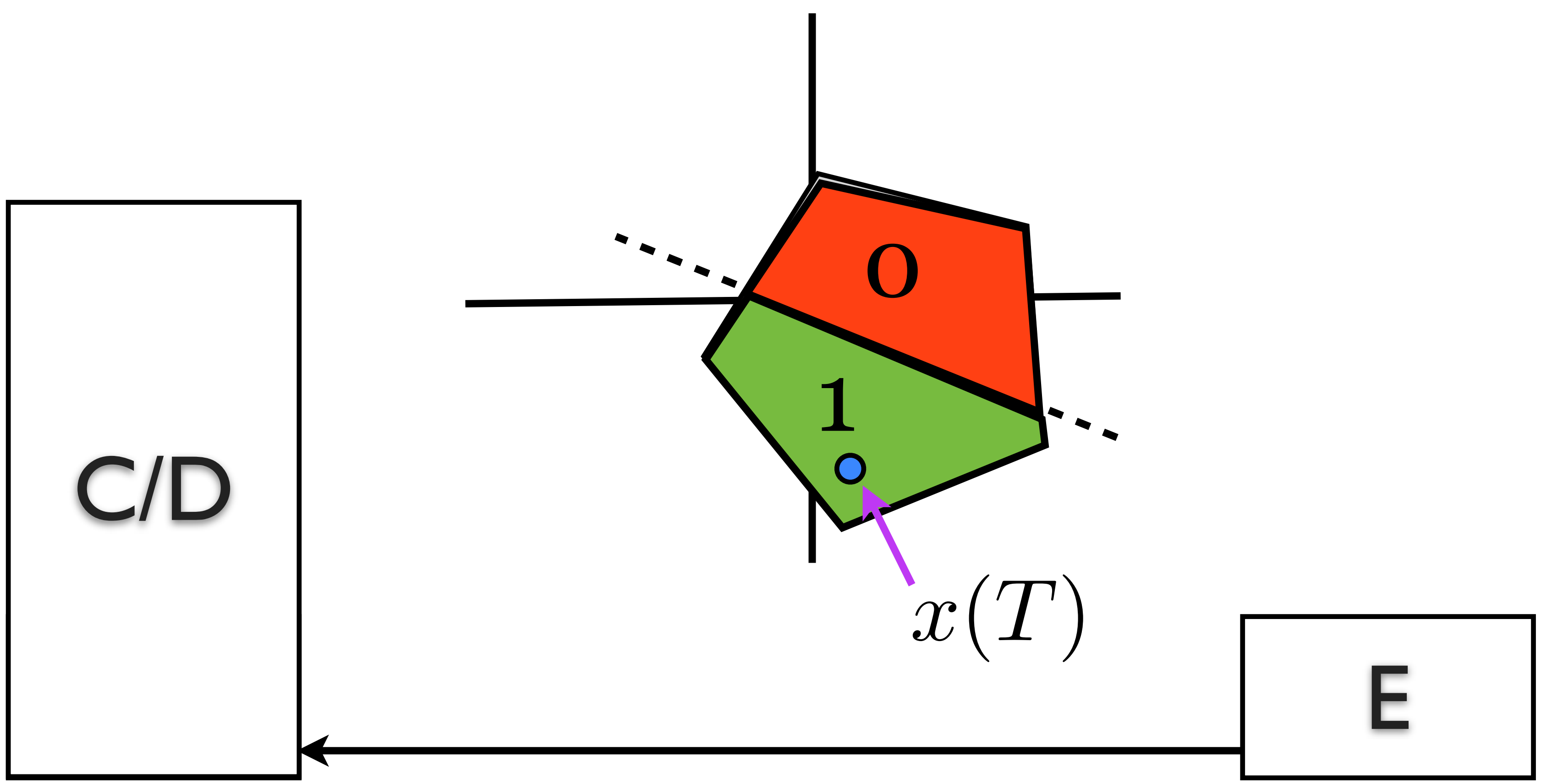

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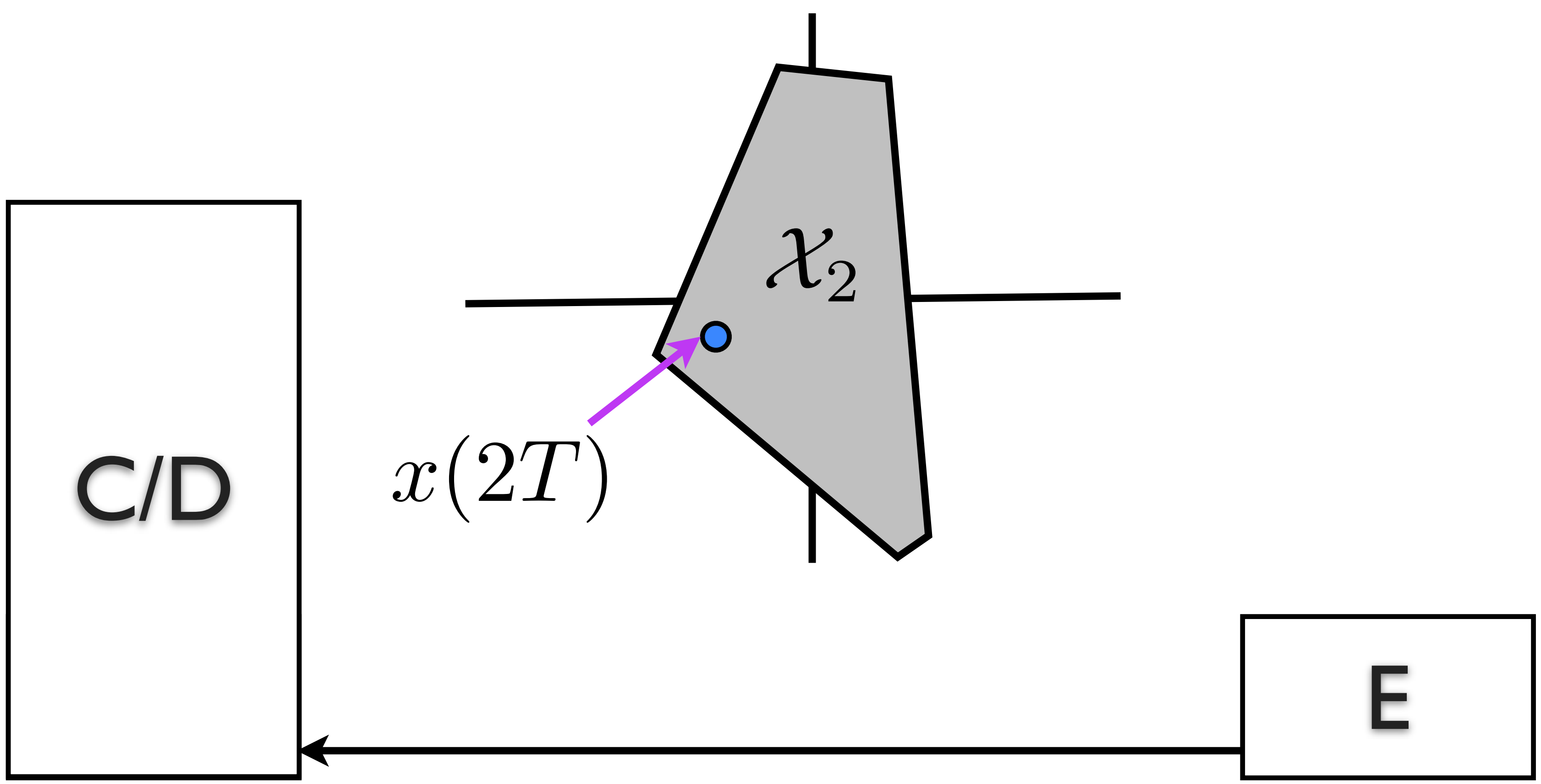

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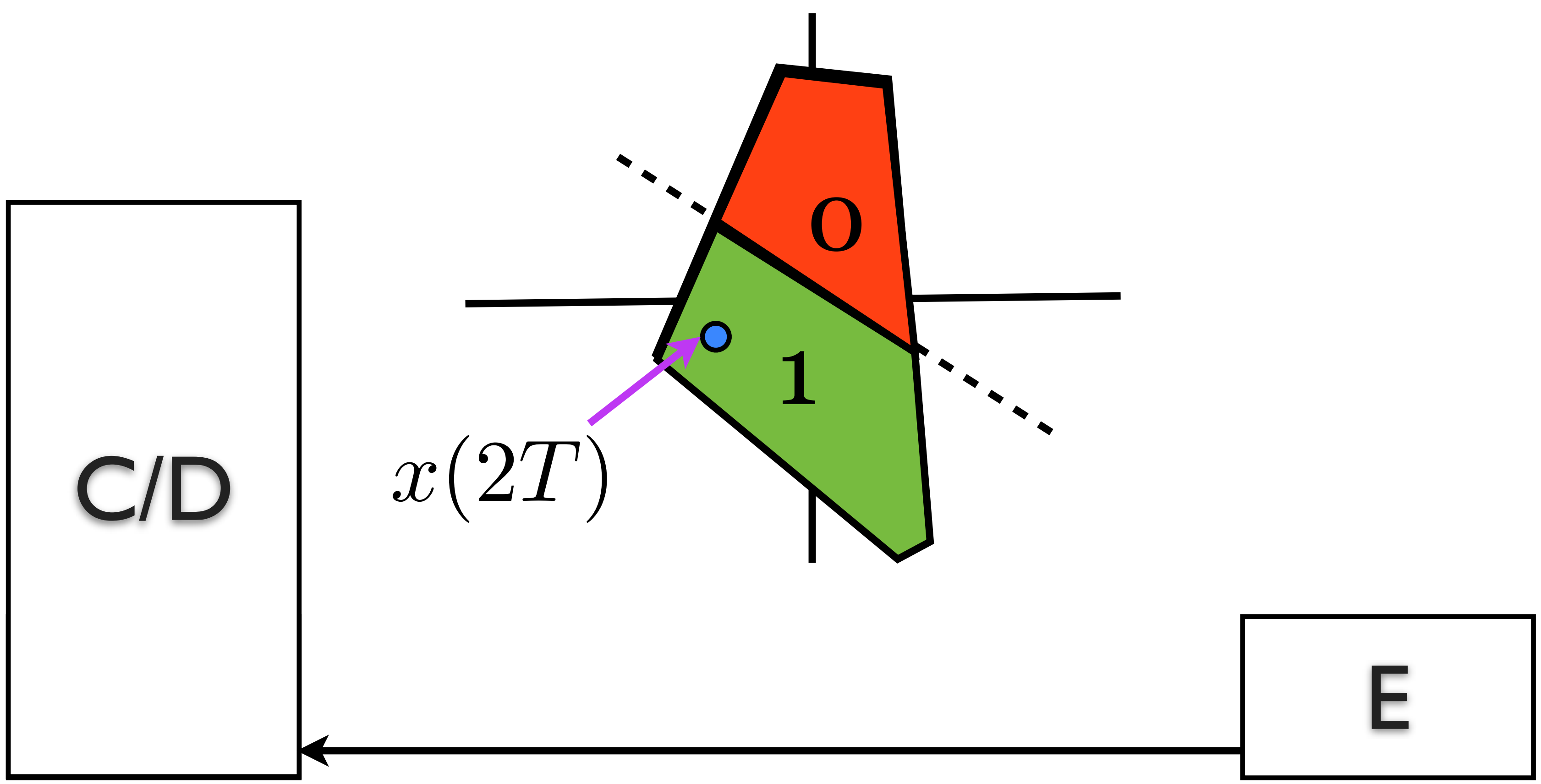

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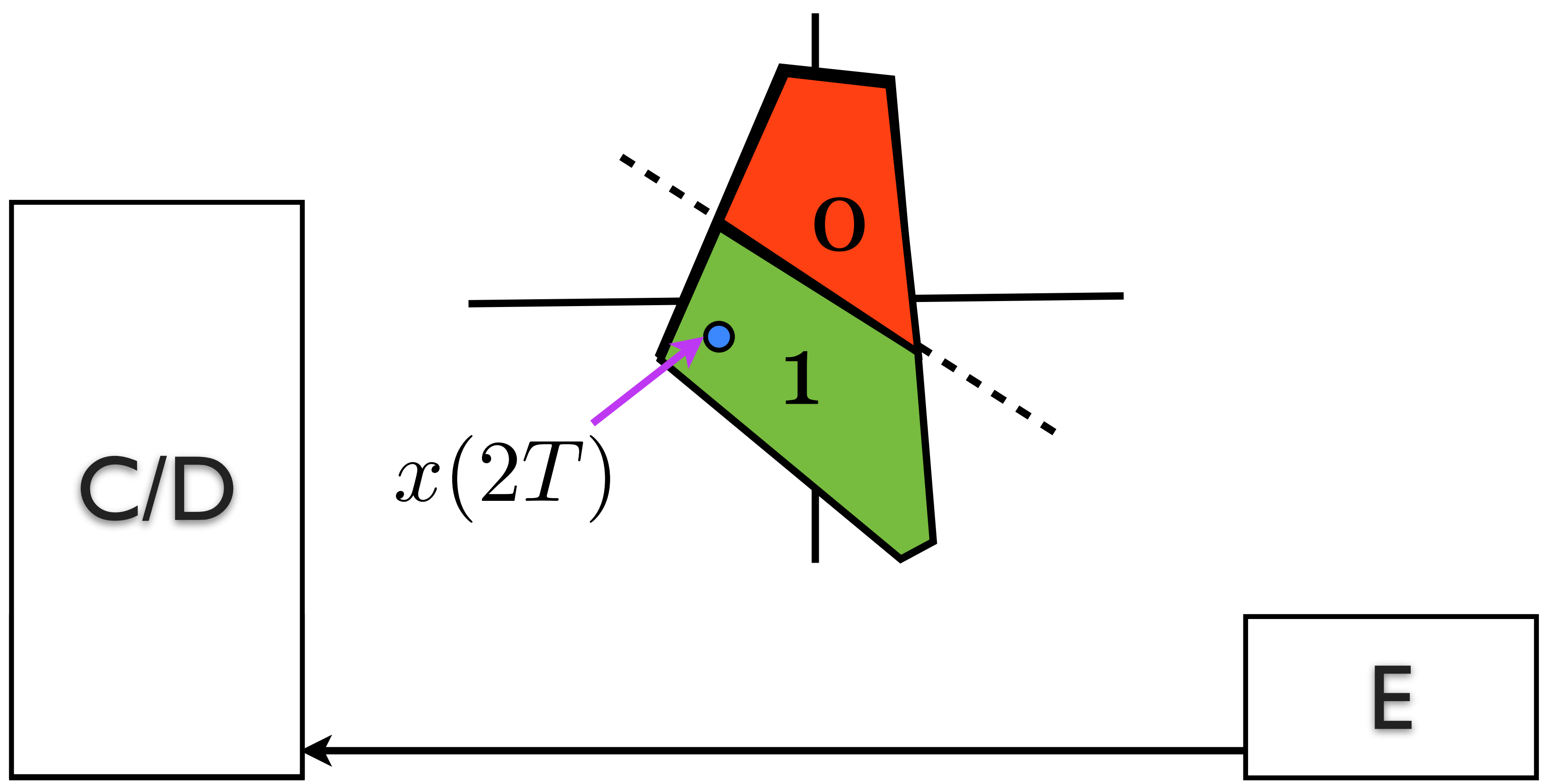

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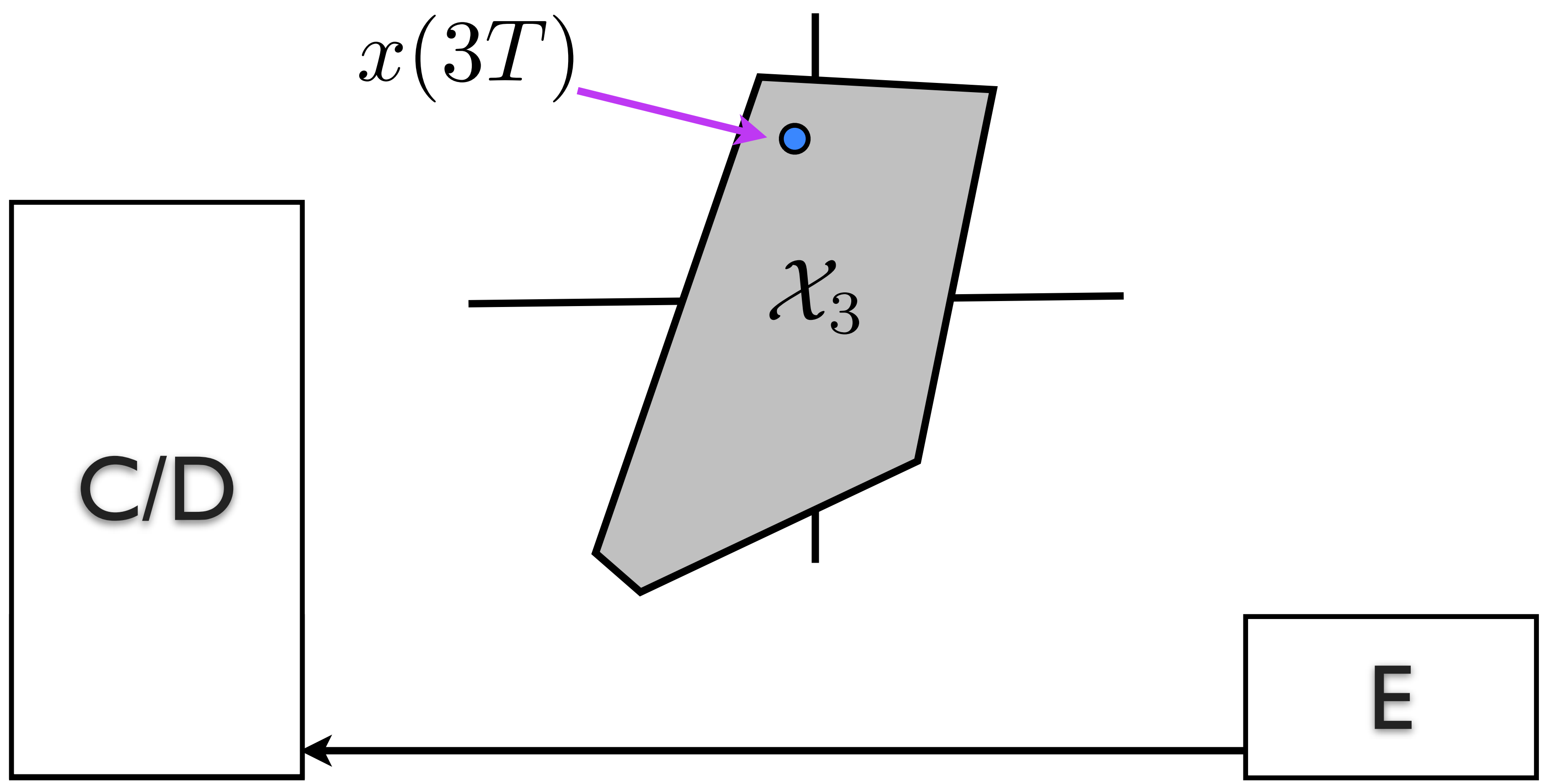

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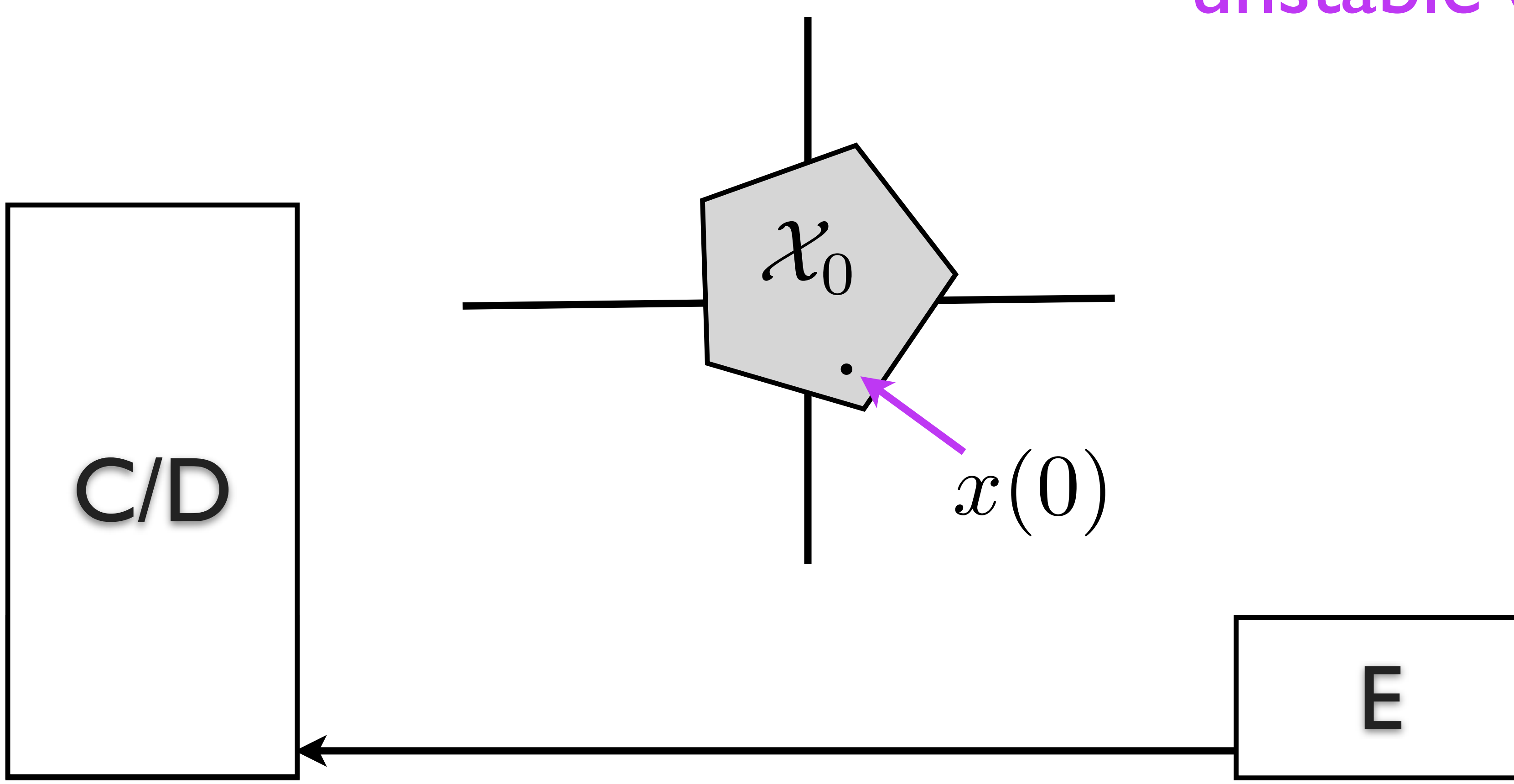

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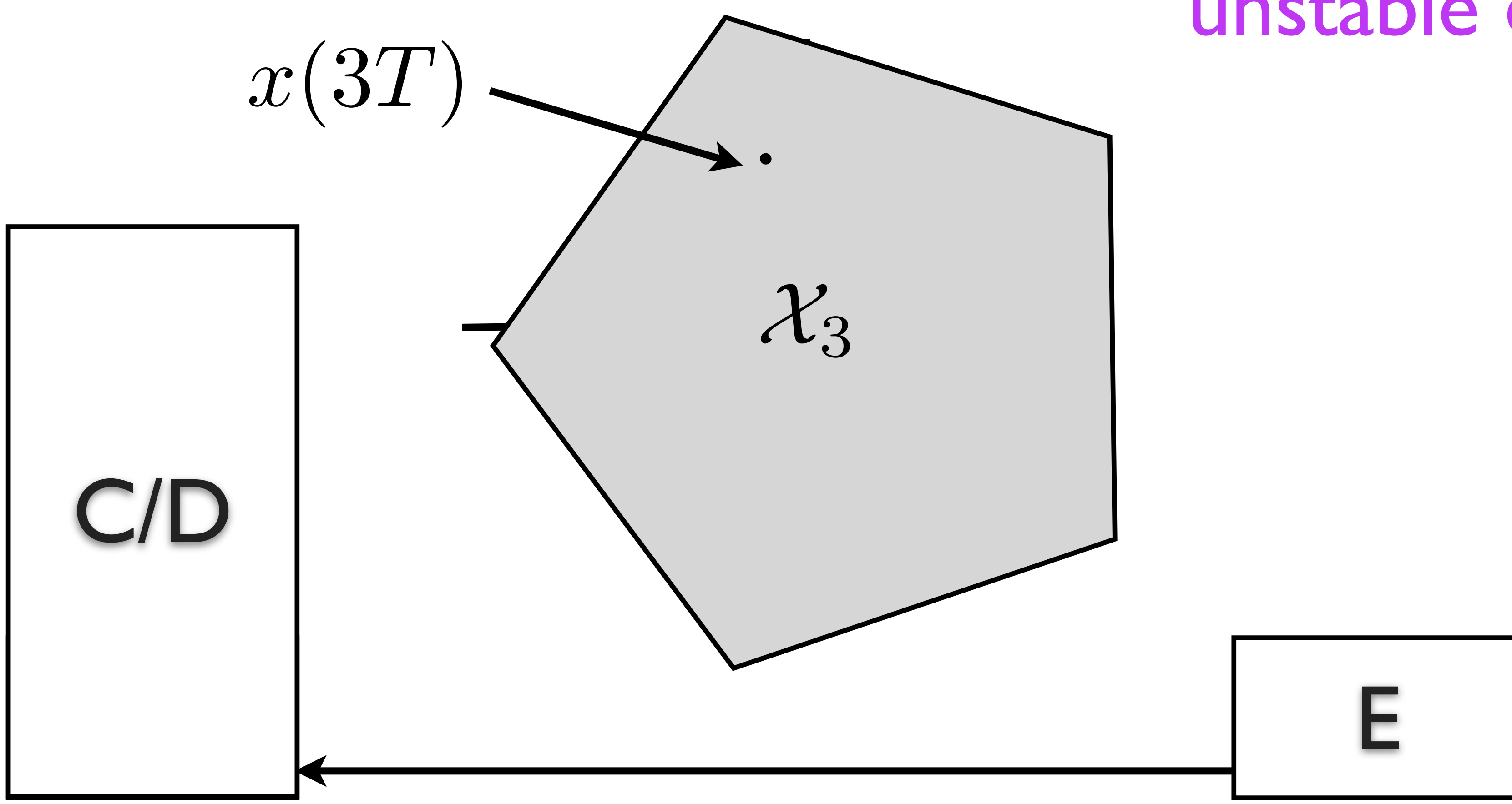

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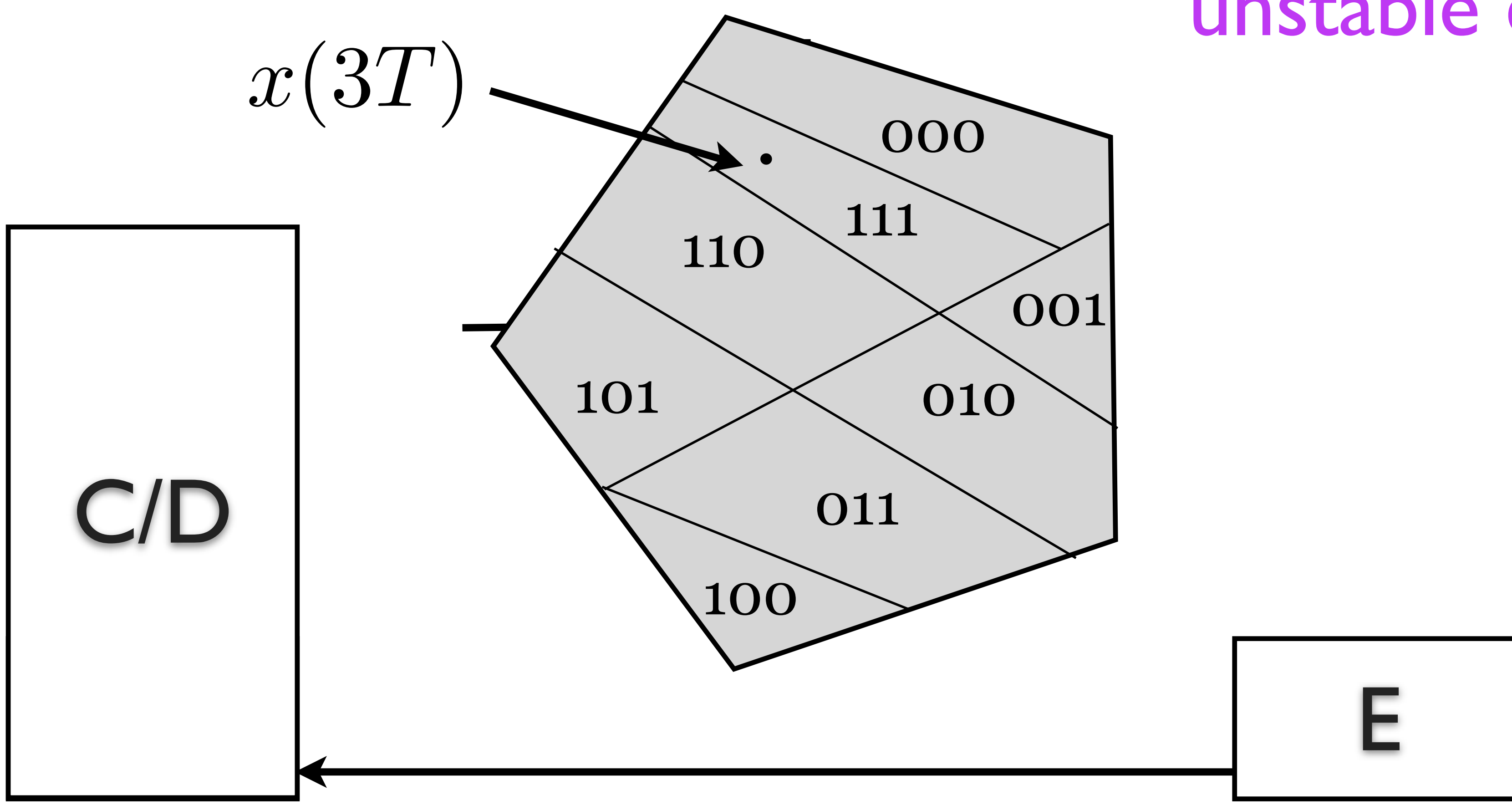
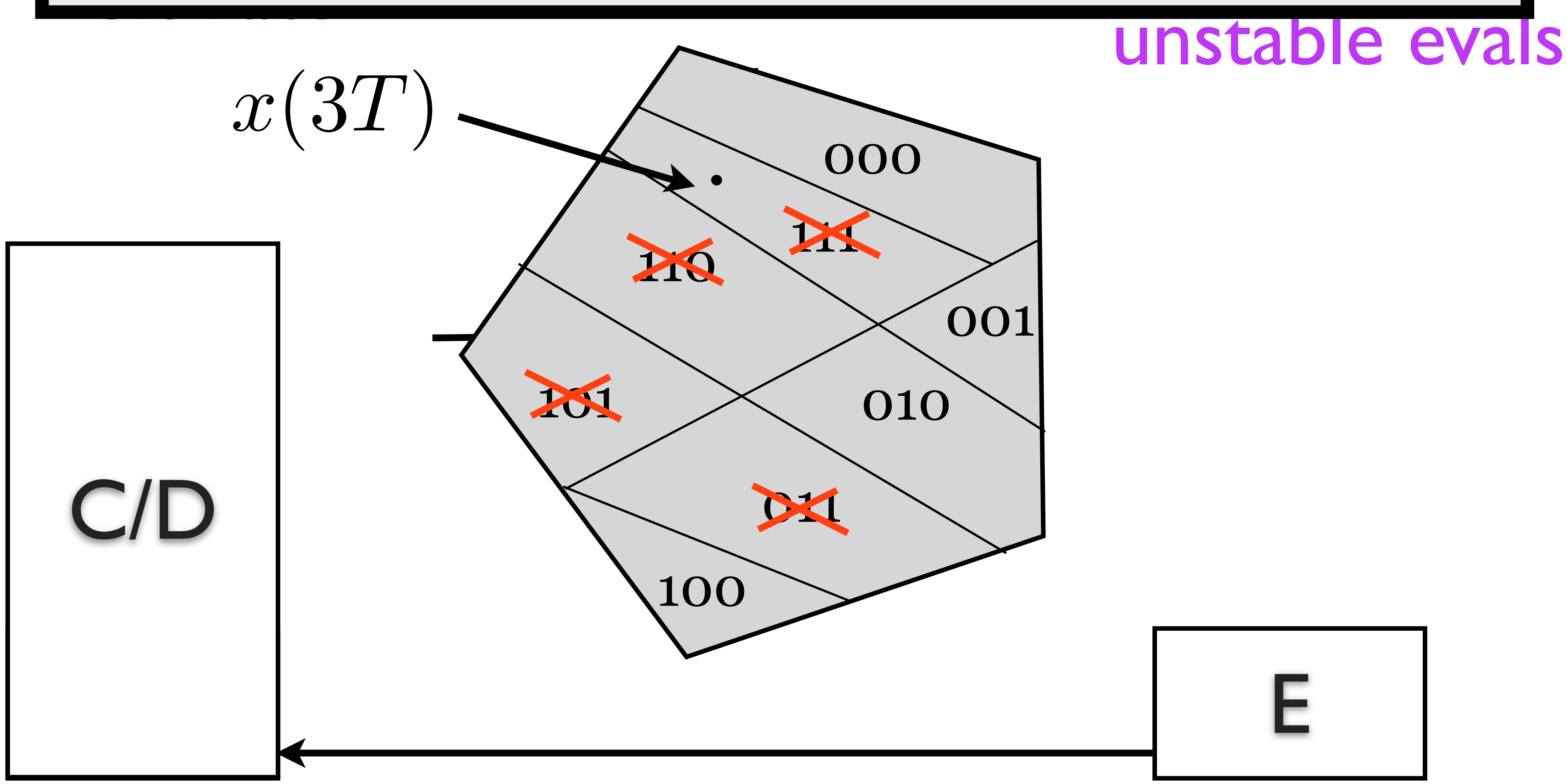


Prior work

What if you can't send arbitrary strings?



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Average communication

- Symbol 0 is free, 1,...,S each consume one unit of communication
  - E.g., the decoder interprets *not transmitting data* as "0"

E.g. γ_{max} = average energy per tx

- Definition: Encoder has *average communication not exceeding* γ_{max} if for any $\{s_k\}$ it may send,

$$\frac{1}{N-M+1}\sum_{k=M}^{N} I_{s_k \neq 0} \leq \gamma_{\max} + O\left(\frac{1}{N-M}\right) \qquad \forall N \geq M \geq 0$$

- Example: {0,0,0,0,1,1,1,1,1,1,...} not exceeding 1
- Example: {0,1,1,0,1,1,0,1,1,...} not exceeding 2/3

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short-term ave: 1

Problem Setup

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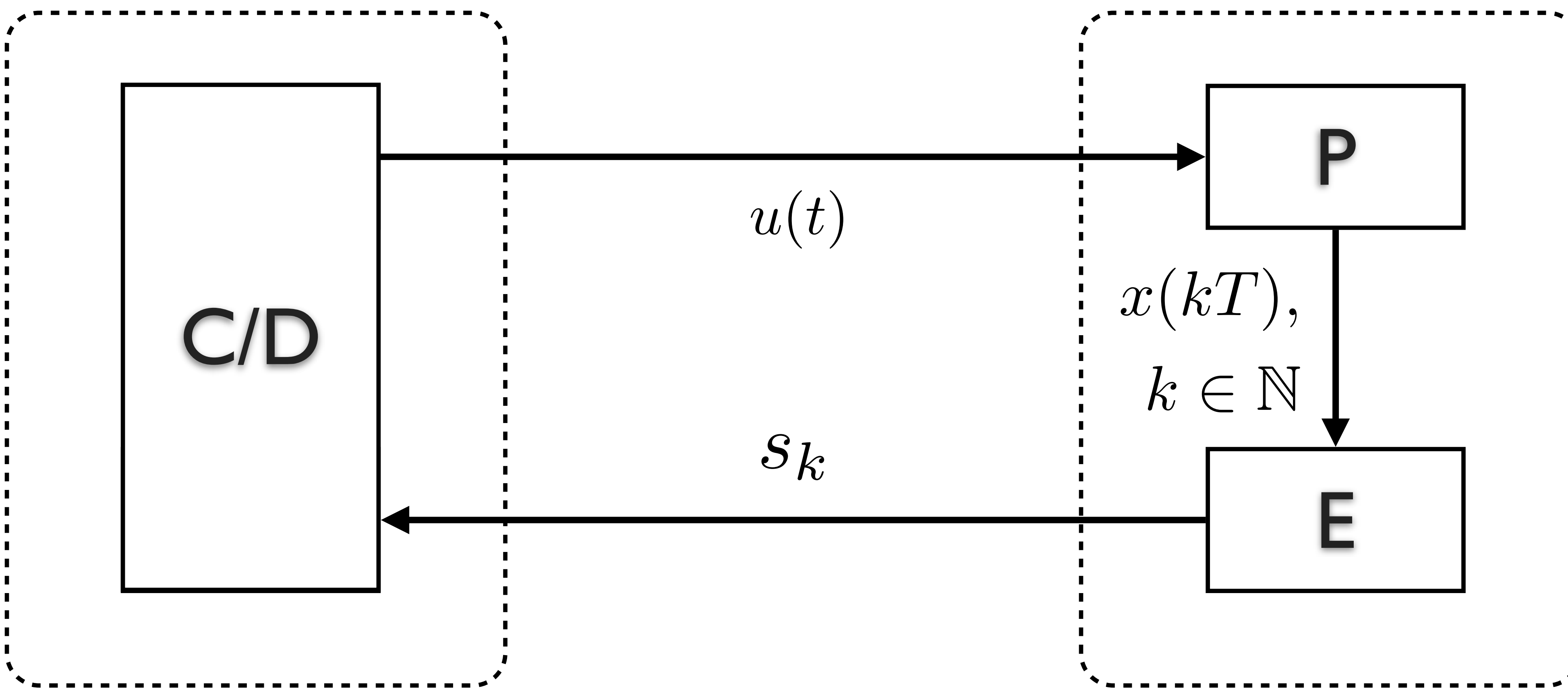

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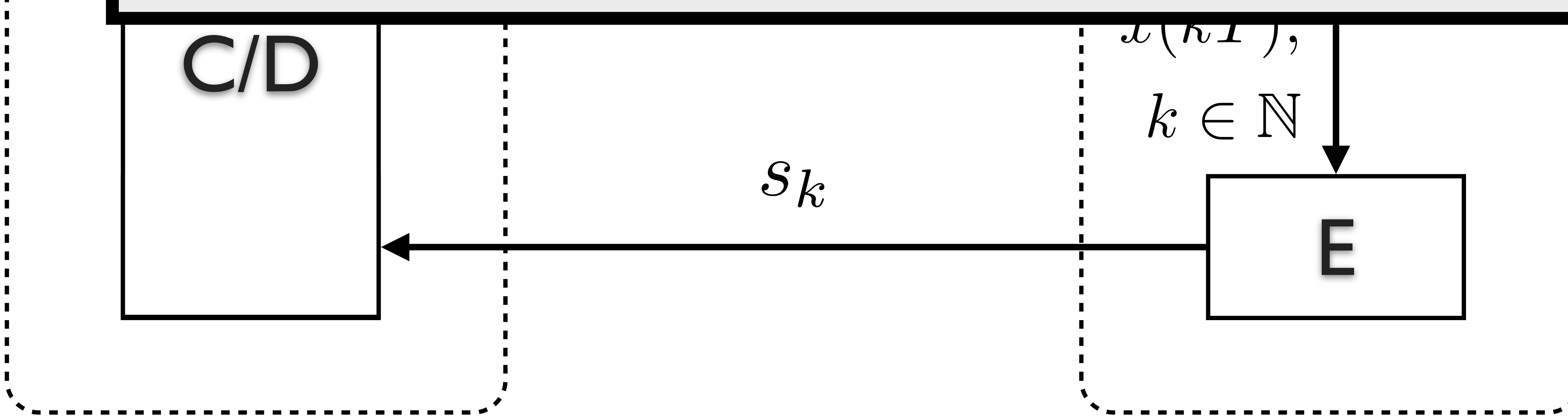
Conditions on S, T, γ, & A to bound state?



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- **Necessary & Sufficient main result**
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Main result

A necessary & sufficient condition on *S*, *T*, γ_{max}, and *A* for a bounding encoder/decoder is

average energy per tx

new

$$r\, \boxed{f(\gamma_{\max}, S)}\, \ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

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bit-rate

$$r := \frac{\log_2(S+1)}{T}$$

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$$r := \frac{\log_2(S+1)}{T}$$

penalty function

$$f(\gamma, S) := \begin{cases} \frac{H(\gamma)+\gamma \log_2 S}{\log_2(S+1)} & 0 \leq \gamma \leq \frac{S}{S+1} \\ 1 & \frac{S}{S+1} < \gamma \leq 1 \end{cases}$$

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base-2 entropy

$$H(p) := -p\log_2(p) - (1-p)\log_2(1-p)$$

Penalty function

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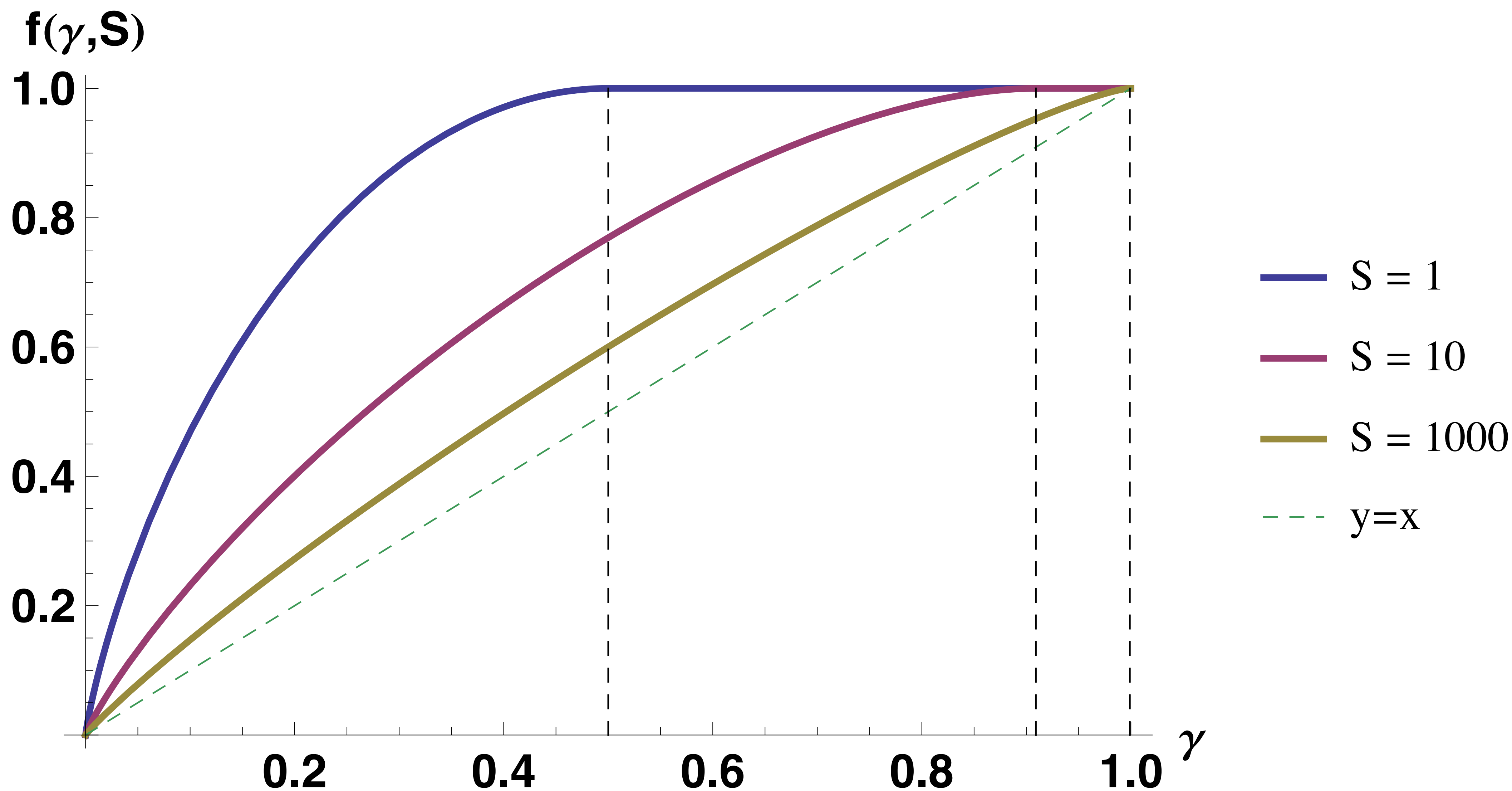

Discussion

A necessary & sufficient condition on *S*, *T*, γ_{max}, and *A* for a bounding encoder/decoder is

average energy per tx

$$r\, f(\gamma_{\max}, S) \ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

$$r_{\min} := \frac{1}{\ln 2} \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

$$\frac{r}{r_{\min}} \geq \frac{1}{f(\gamma_{\max}, S)}$$

bit-rate penalty factor

Discussion

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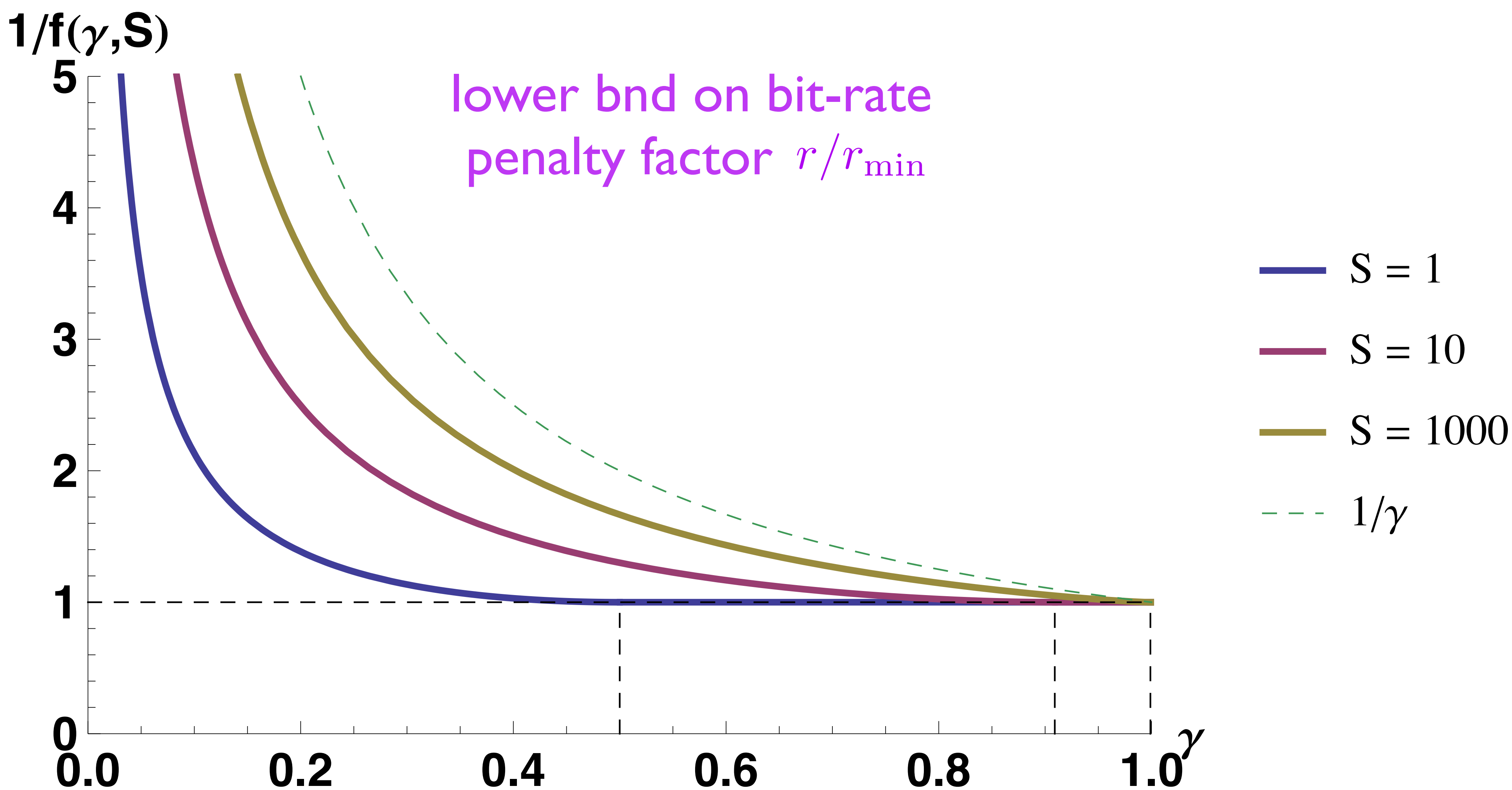

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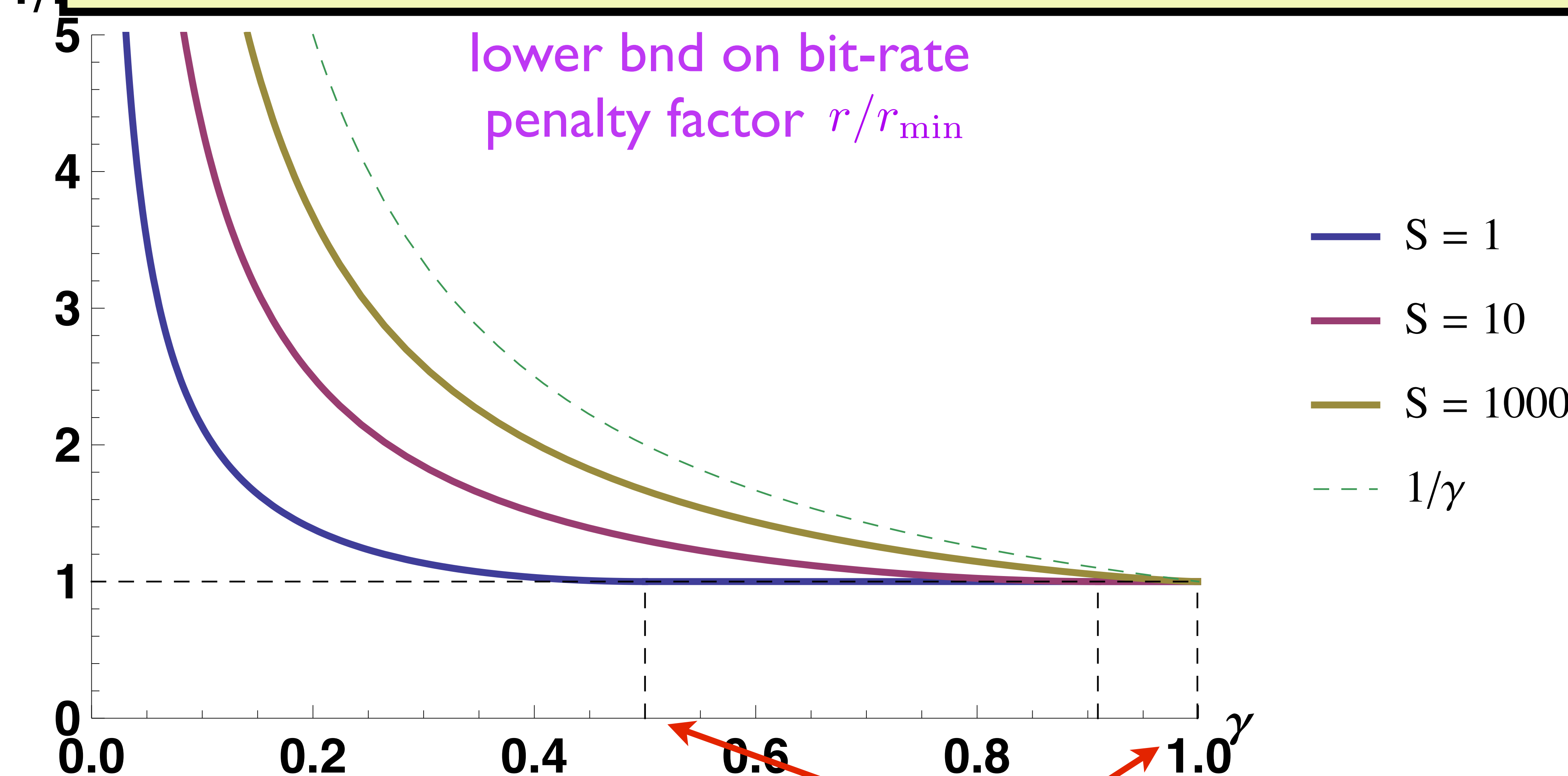
Observation 1: γ=S/(S+1) is "just as good" as γ=1

lower bnd on bit-rate penalty factor $r/r_{\min}$



caveat: very different encoders!

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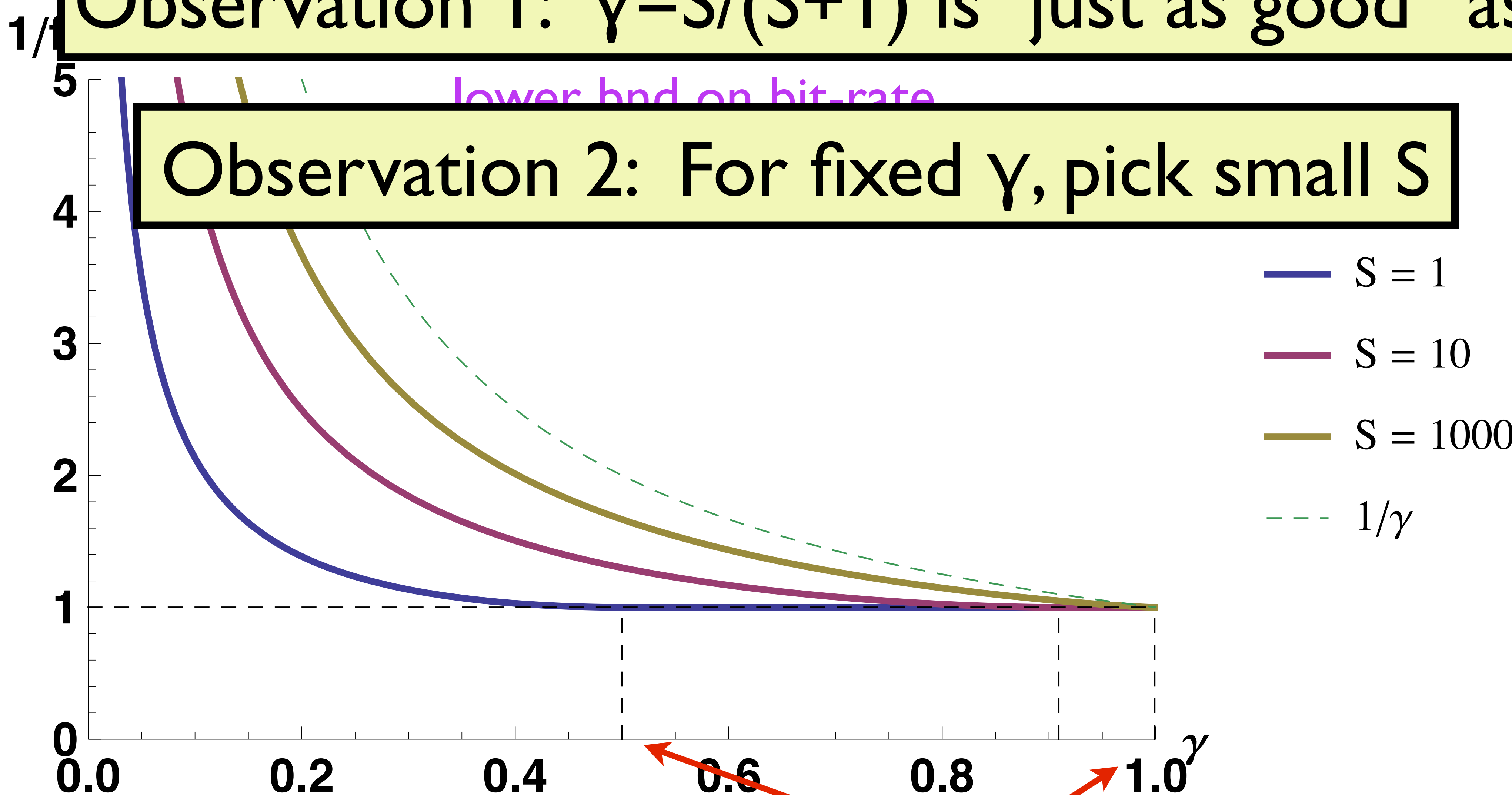
Observation 1: γ=S/(S+1) is "just as good" as γ=1

Observation 2: For fixed γ, pick small S



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Discussion

Average rate of nonfrees per second: γ/T

Let γ be given. For any $\epsilon > 0,$

1. Pick large T to make $\gamma/T < \epsilon$

2. $r\, f(\gamma, S) \ln 2$ monot. inc. in S

3. Pick large S to satisfy

$$r\, f(\gamma_{\max}, S) \ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

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Observation 3: Can make #nf/sec arbitrarily small

(at the cost of performance & complexity...)

Discussion

Average rate of nonfrees per second: γ/T

Let S be given. For any $\epsilon > 0$,

1. Define sequences

$$\gamma_k := e^{-k}, \qquad T_k := e^{-k}\sqrt{k}, \qquad k \in \mathbb{N}_{\geq 0}$$

2. Then $\gamma_k \to 0, \quad T_k \to 0, \quad \gamma_k / T_k \to 0$

but also $\dfrac{H(\gamma_k)}{T_k} = \dfrac{-\gamma_k \ln \gamma_k}{T_k} + O(\gamma_k) \to \infty$

3. So $r_k\, f(\gamma_k, S) \ln 2 \to \infty, \qquad \forall S \in \mathbb{N}_{>0}$

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Observation 3b: Can make #nf/sec arbitrarily small

(at the cost of requiring a precise clock...)

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Average rate of nonfrees per second: γ/T

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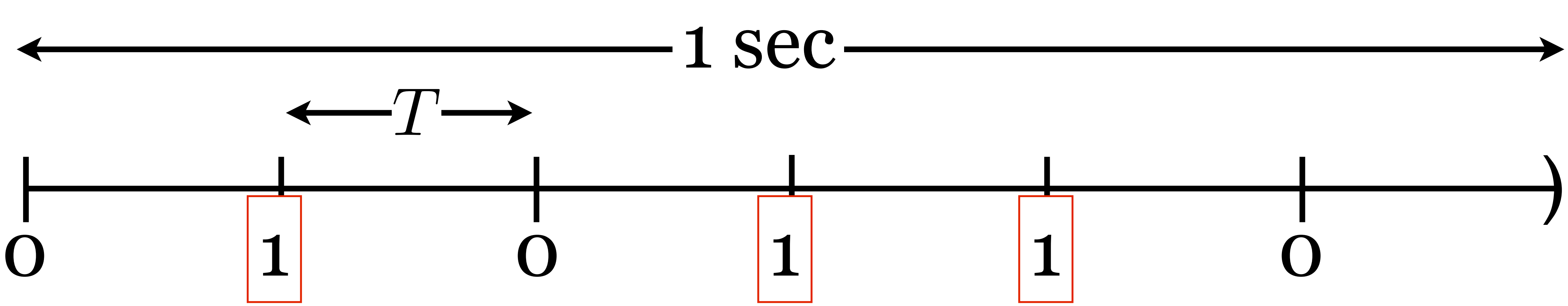
Example: 3 nfs/sec, T=1/6



$\binom{6}{3}$ sequences, $\log_2 \binom{6}{3}$ bits/sec

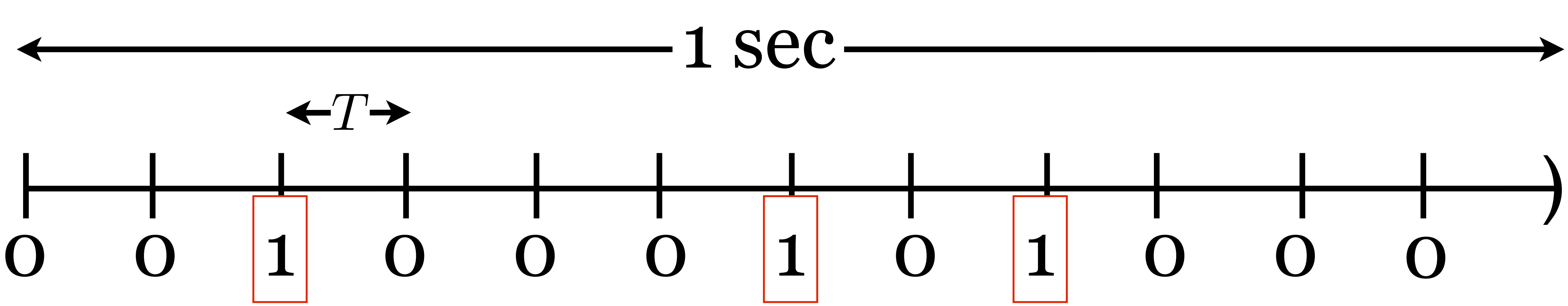
Discussion

Average rate of nonfrees per second: γ/T

Observation 3b: Can make #nf/sec arbitrarily small

(at the cost of requiring a precise clock...)

Example: 3 nfs/sec, T=1/12



$\binom{12}{3}$ sequences, $\log_2 \binom{12}{3}$ bits/sec

Discussion

$$\underbrace{r\ \ln 2}_{} \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

bit-rate
(nats/sec)

unstable evals

Discussion

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bit-rate (nats/sec)

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$$\left.\begin{matrix} \frac{H(\gamma)+\gamma\ln S}{T} & 0 \leq \gamma \leq \frac{S}{S+1} \\ \frac{\ln(S+1)}{T} & \frac{S}{S+1} < \gamma \leq 1 \end{matrix}\right\} = r\,\boxed{f(\gamma, S)}\ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

Discussion

$$\underbrace{r\ \ln 2}_{\text{bit-rate (nats/sec)}} \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

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new

entropy of RV taking values {0,...,S} w.p.

$$p(x) = \begin{cases} 1-\gamma & x = 0 \\ \gamma/S & x = 1, \ldots, S \end{cases}$$

$$H(X) = H(\gamma) + \gamma \ln S$$

Discussion

$$\underbrace{r \ln 2}_{} \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

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$$p(x) \quad \begin{cases} 1-\gamma & x=0 \\ \ldots \end{cases}$$

Observation 4: as before, bit-rate must exceed rate of state explosion

$$H(X) = H(\gamma) + \gamma \ln S$$

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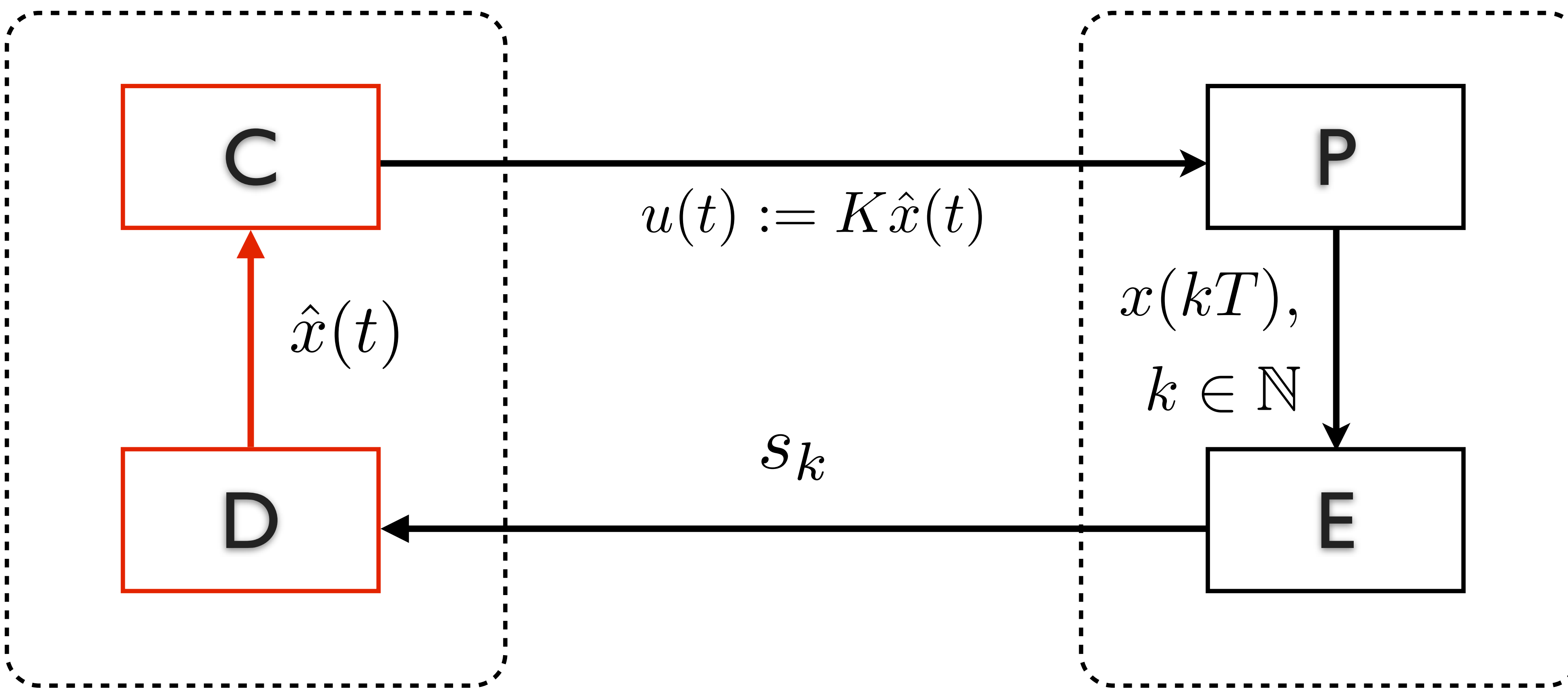
- **Event-based encoding**

Event-based Encoding

- "Emulation-based" controller:
  - $u(t) := Kx(t)$ stabilizes original process
  - (Assume A-BK is Hurwitz)
- "Event-based" encoder/decoder pair

  0 : absence of an event



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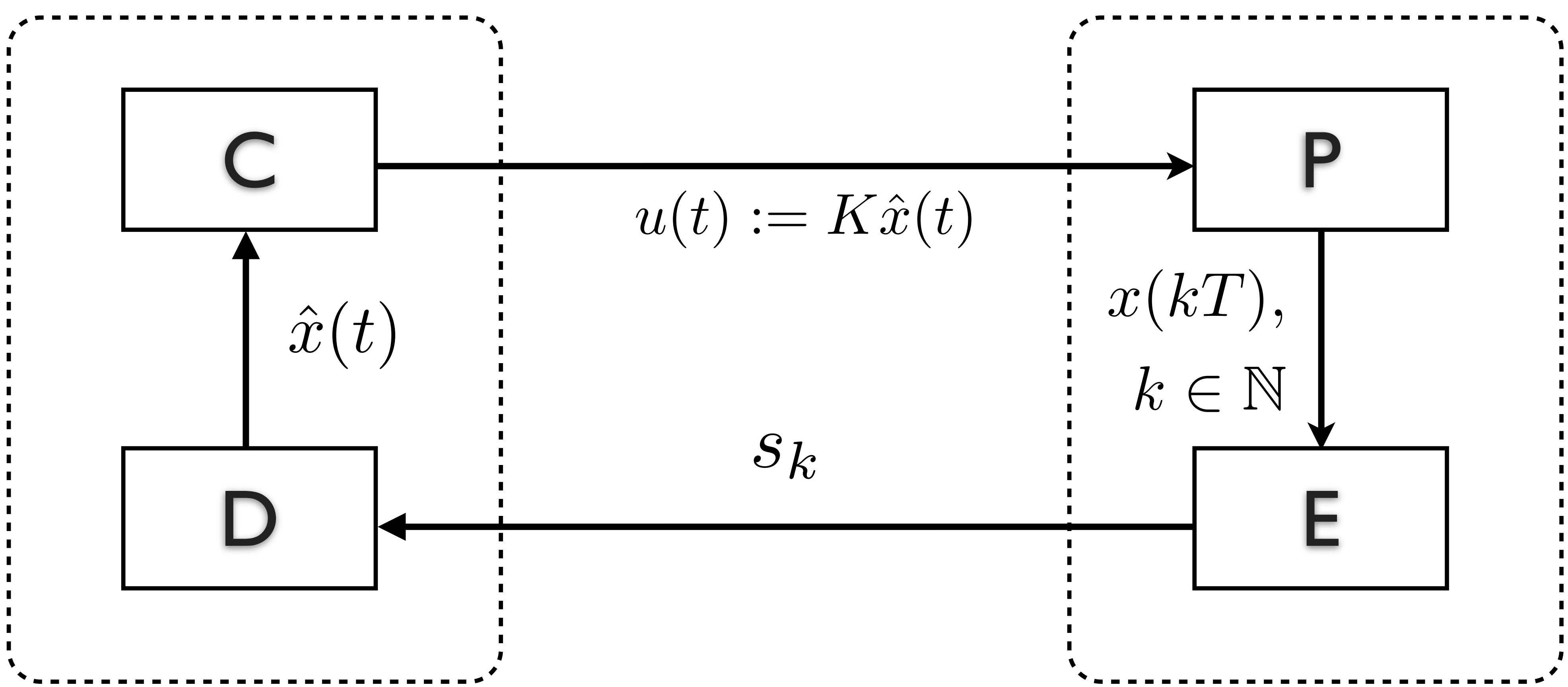

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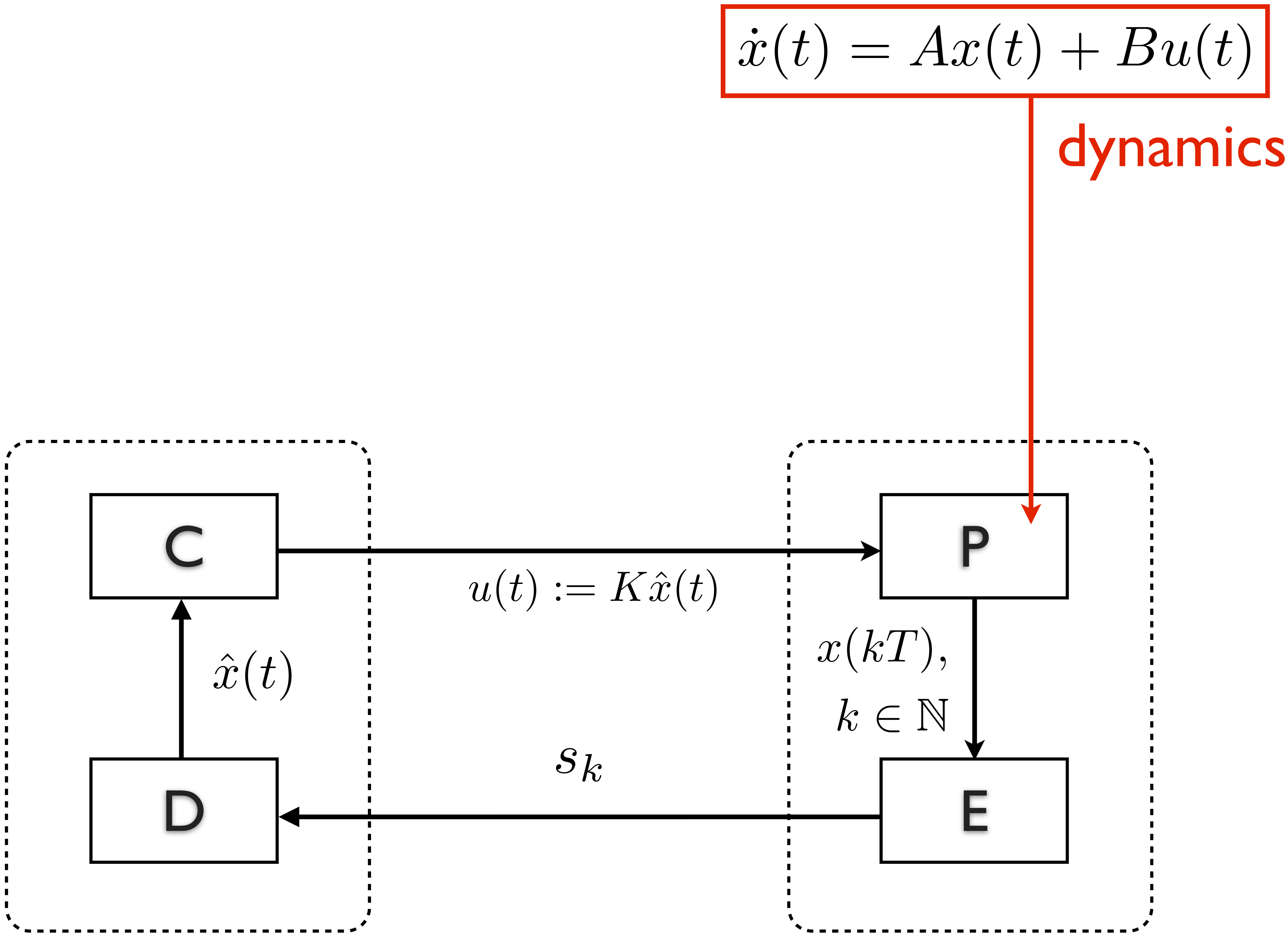

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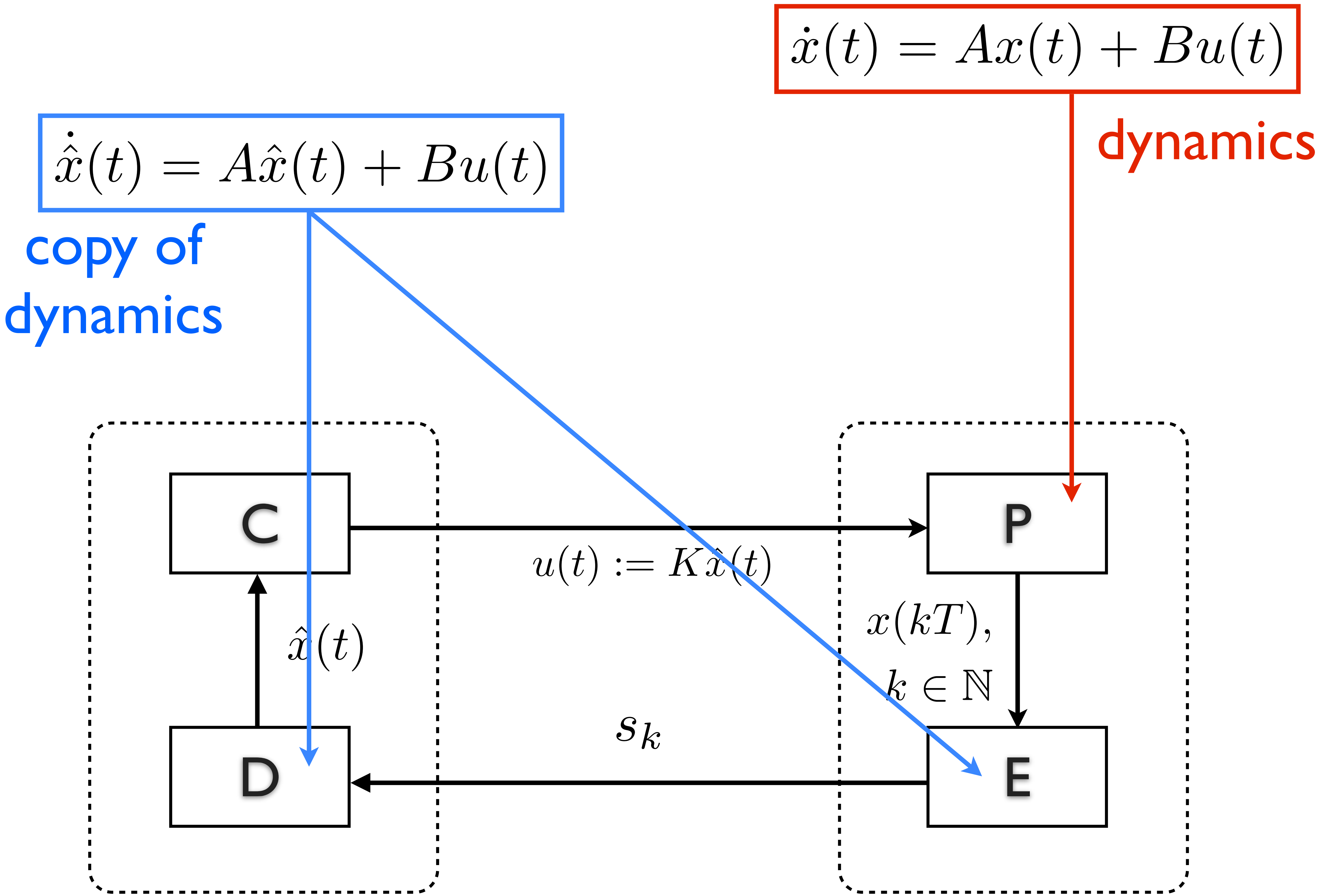

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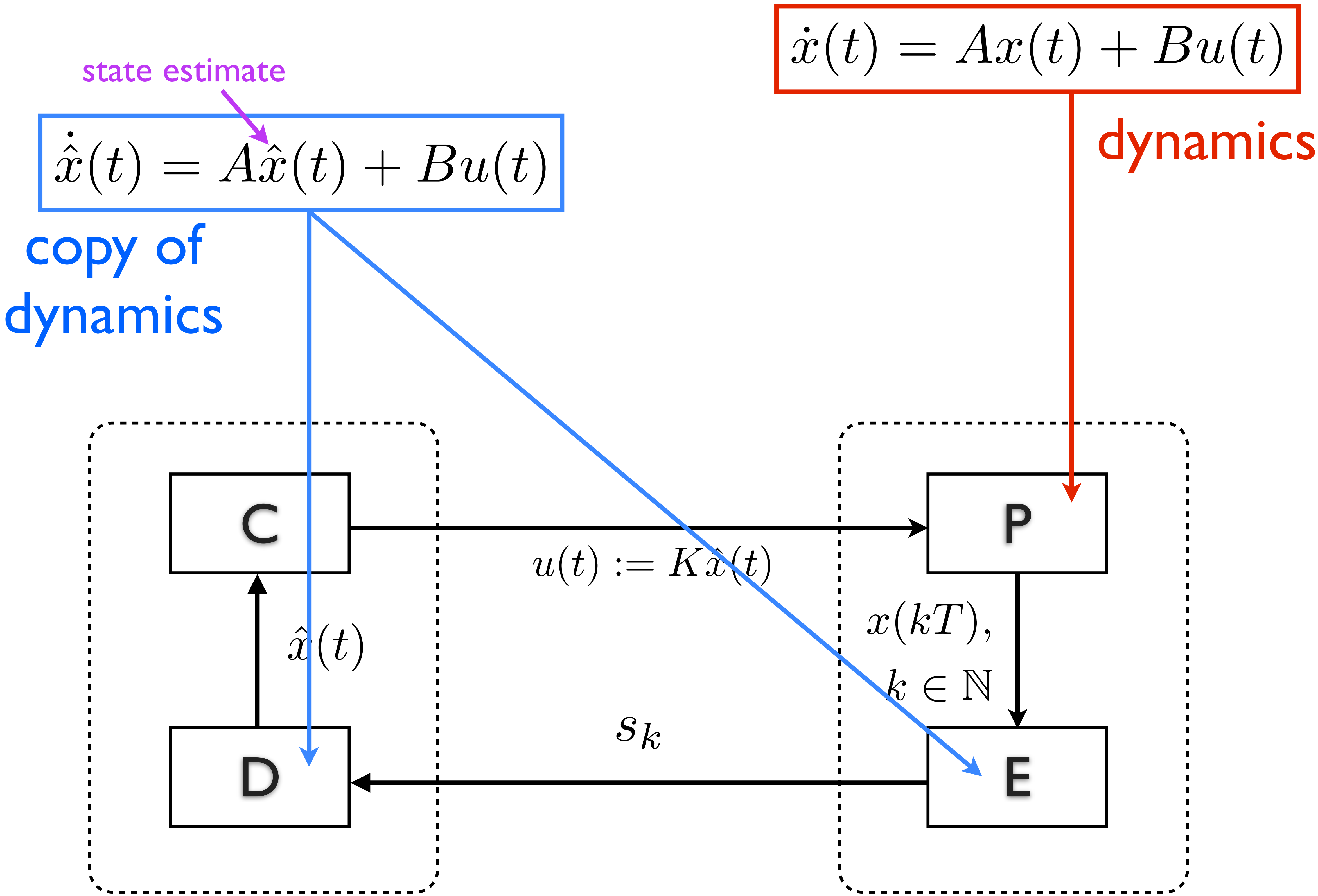

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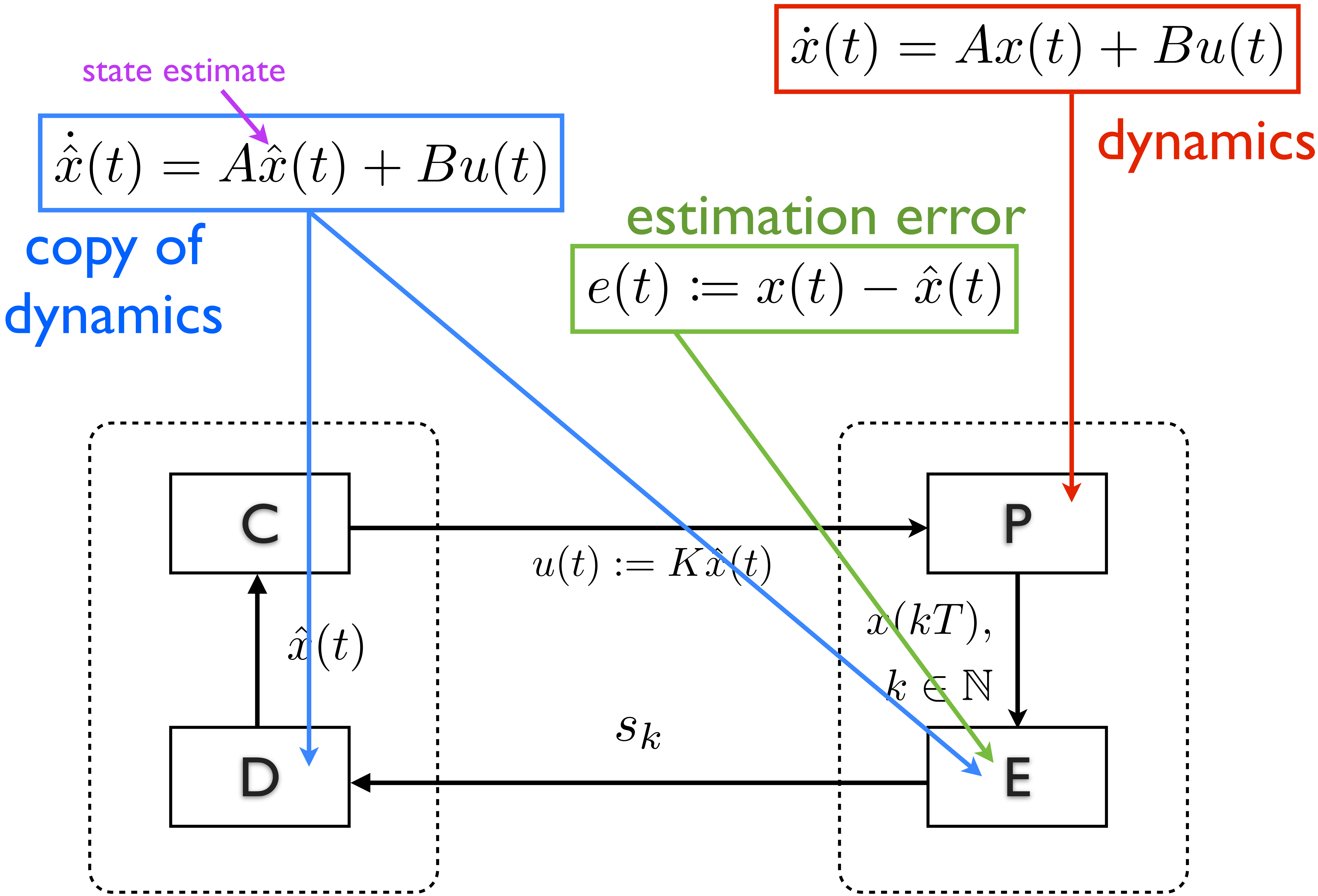

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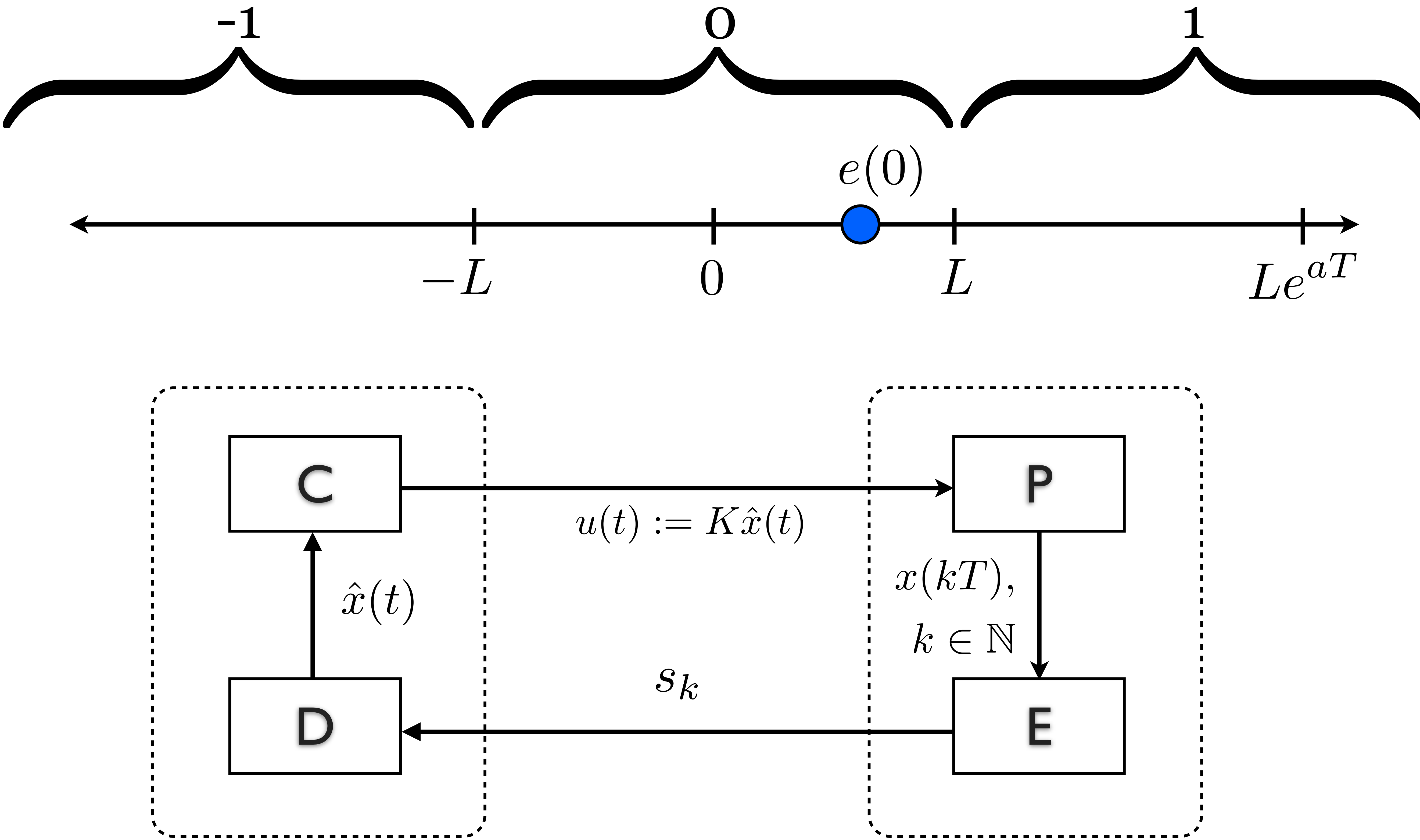
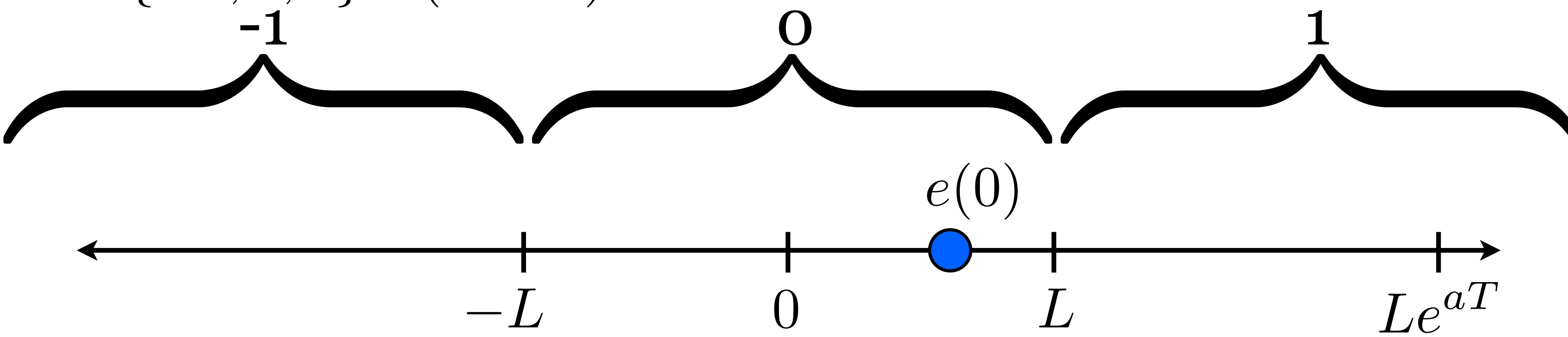
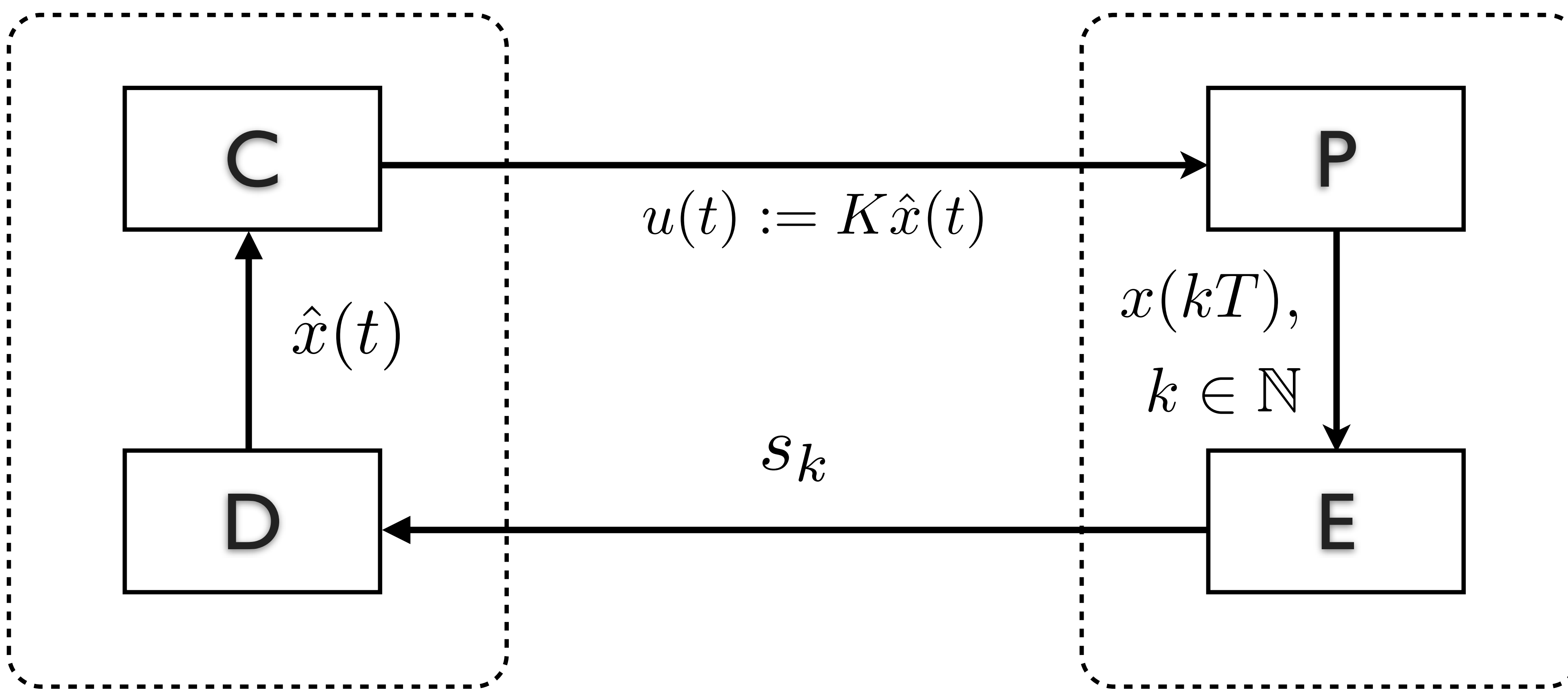


Event-based Encoding

$\mathcal{A} := \{-1, 0, 1\} \quad (S = 2)$





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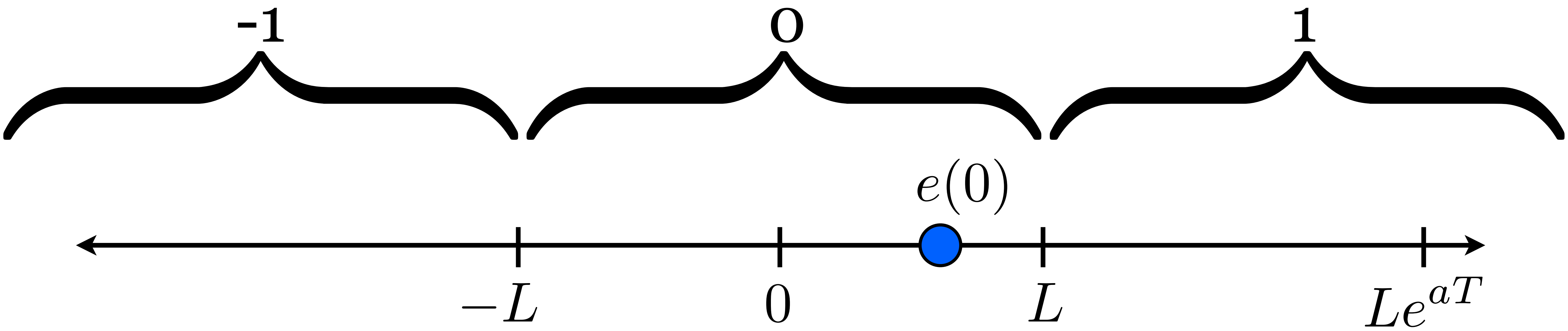

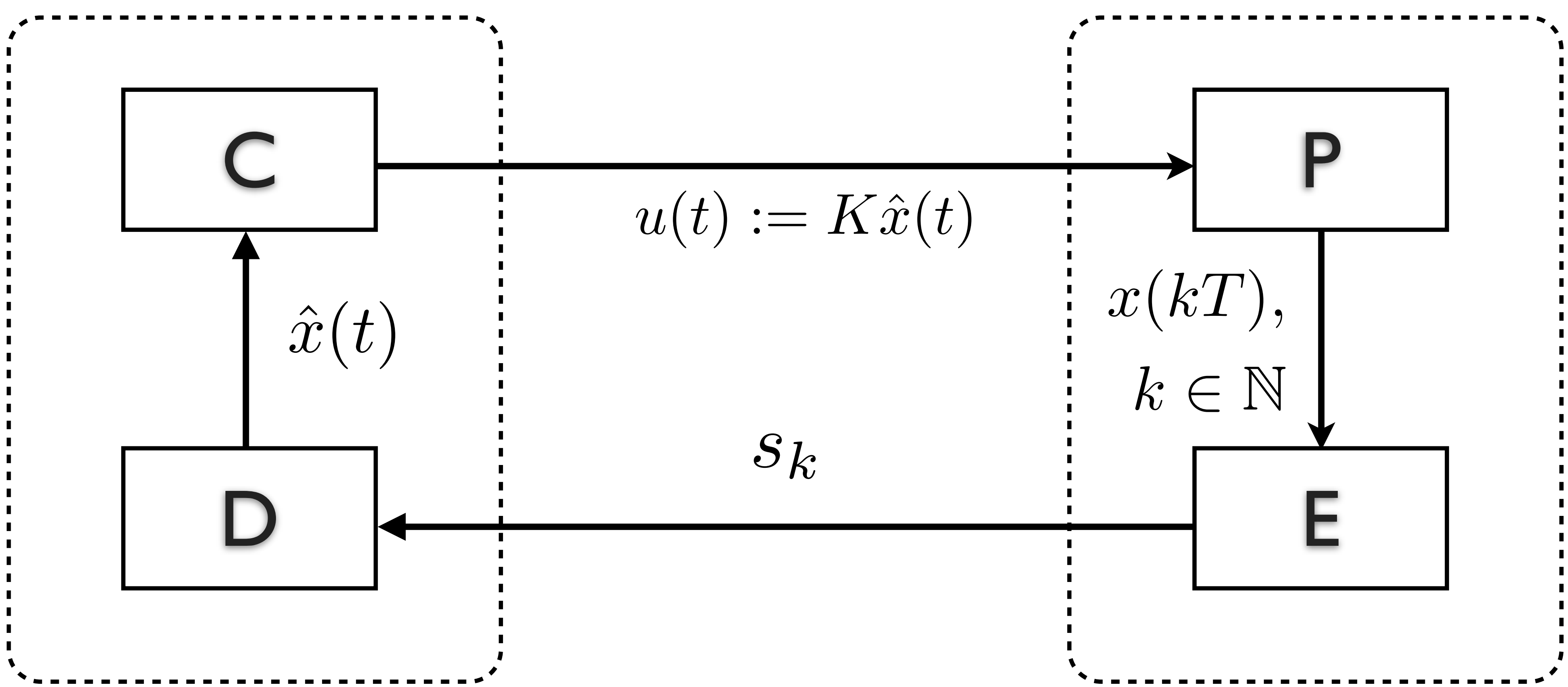

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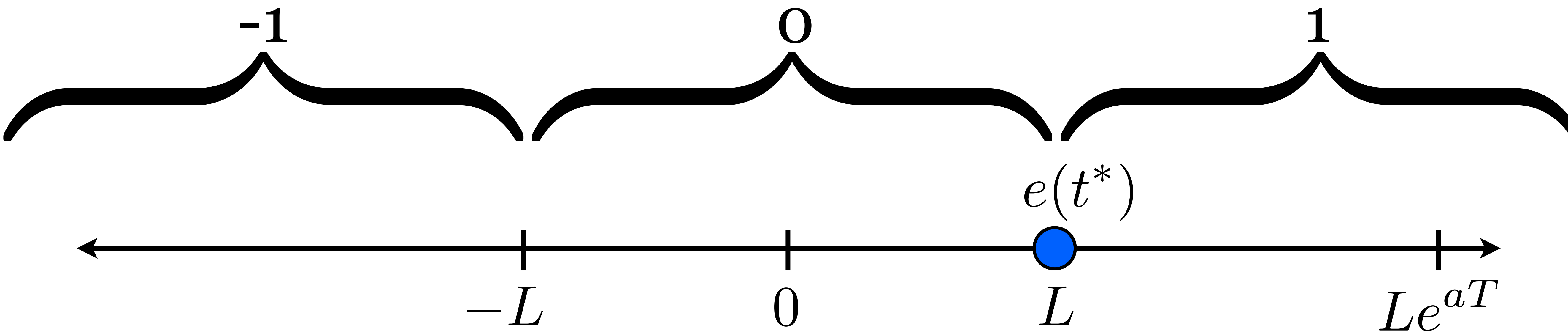

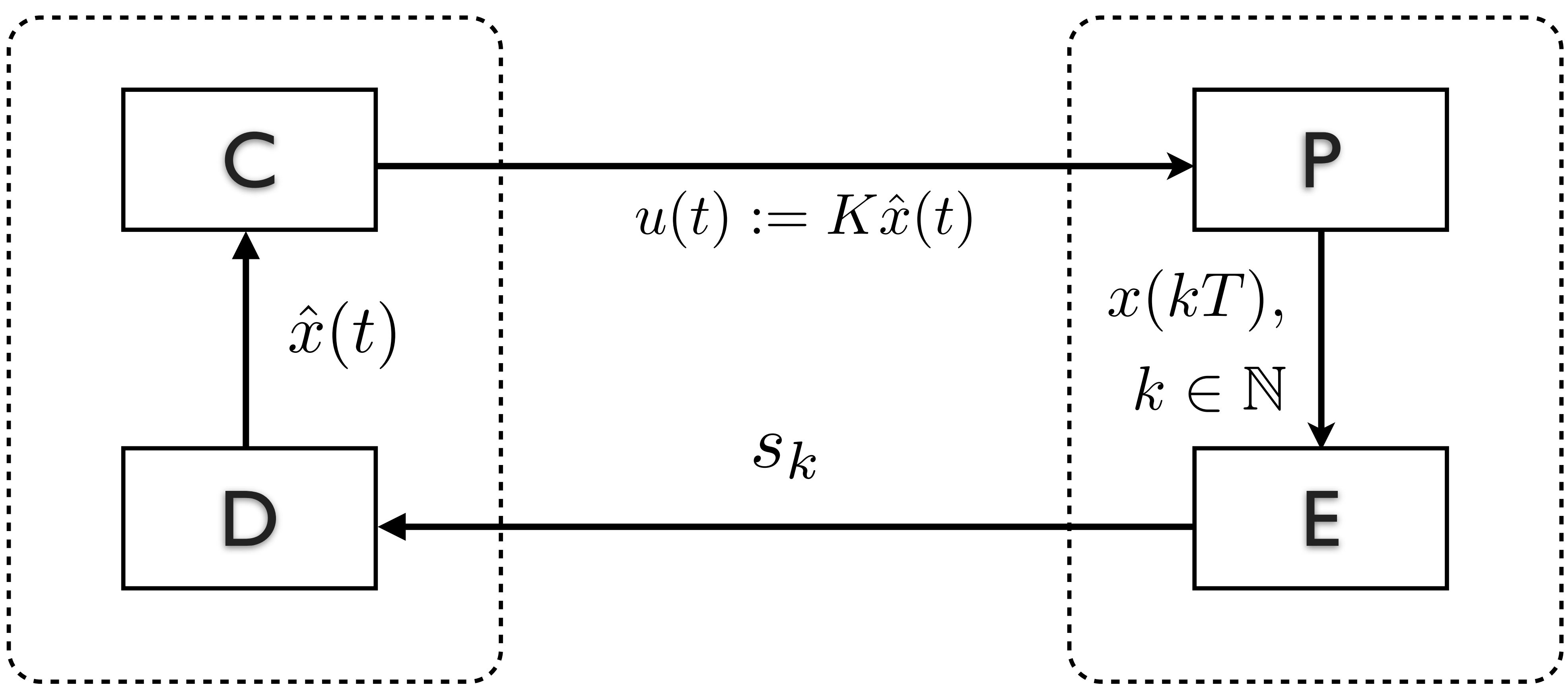

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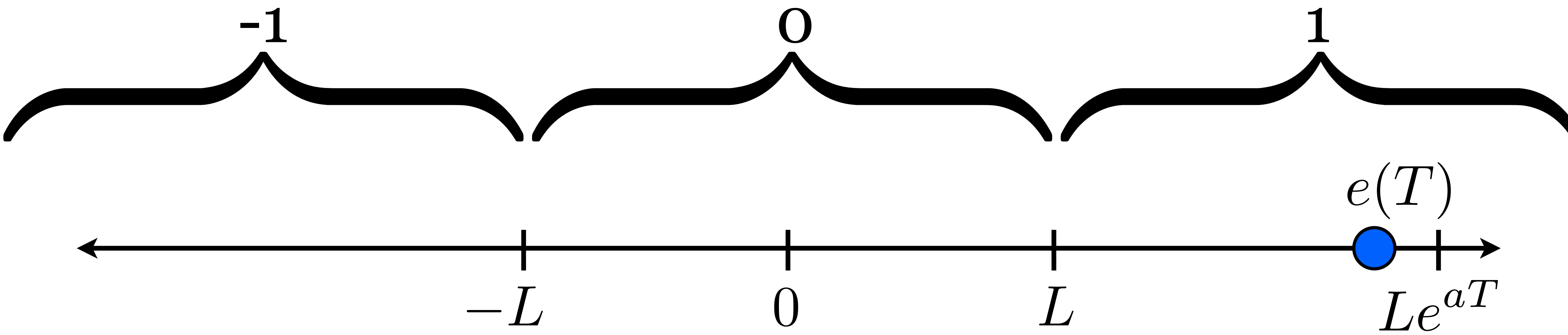

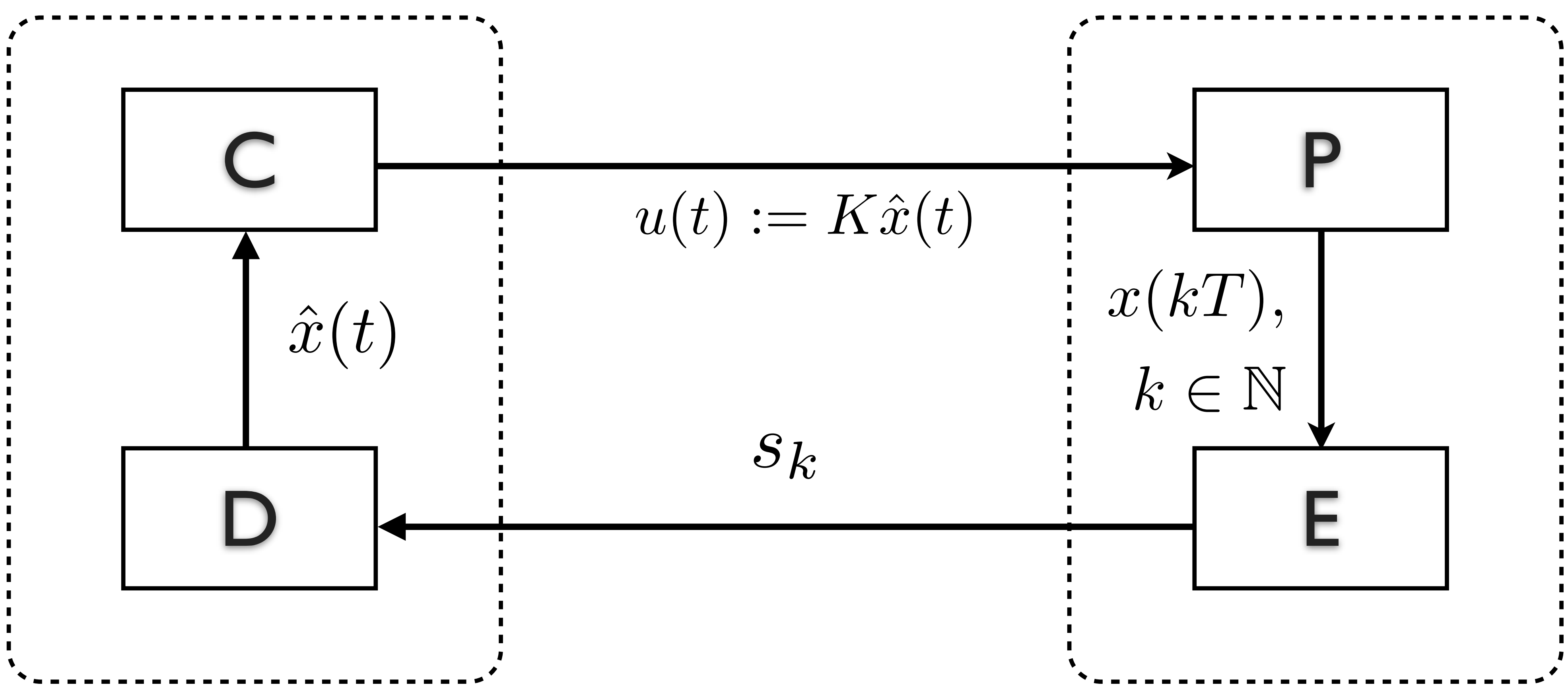

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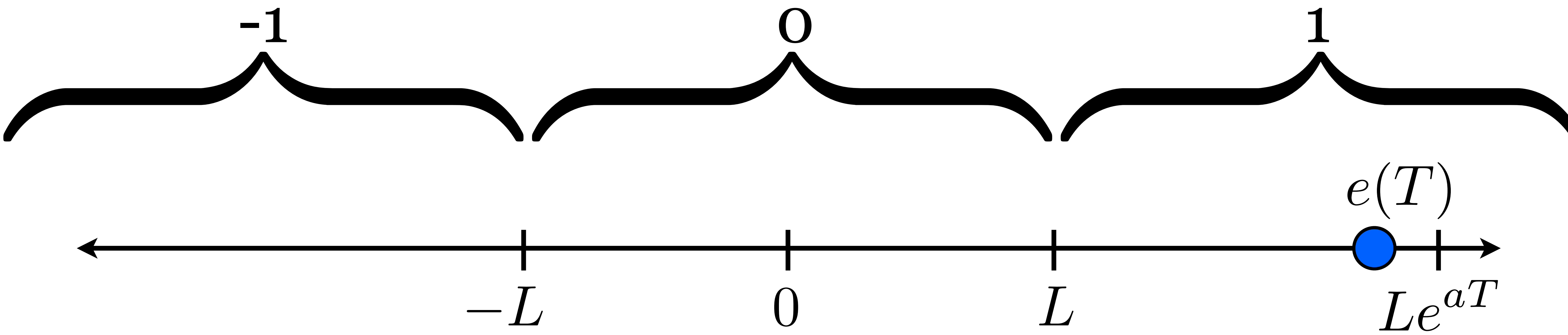

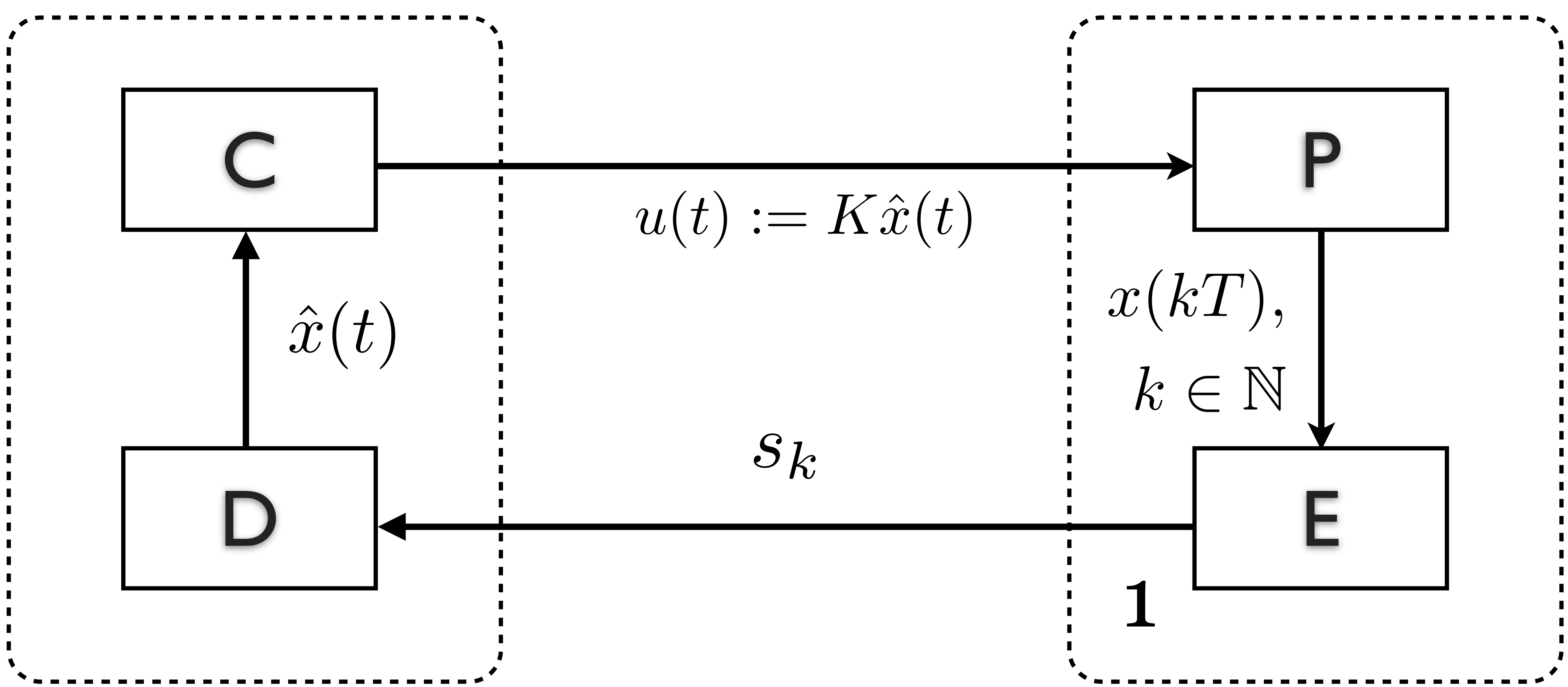

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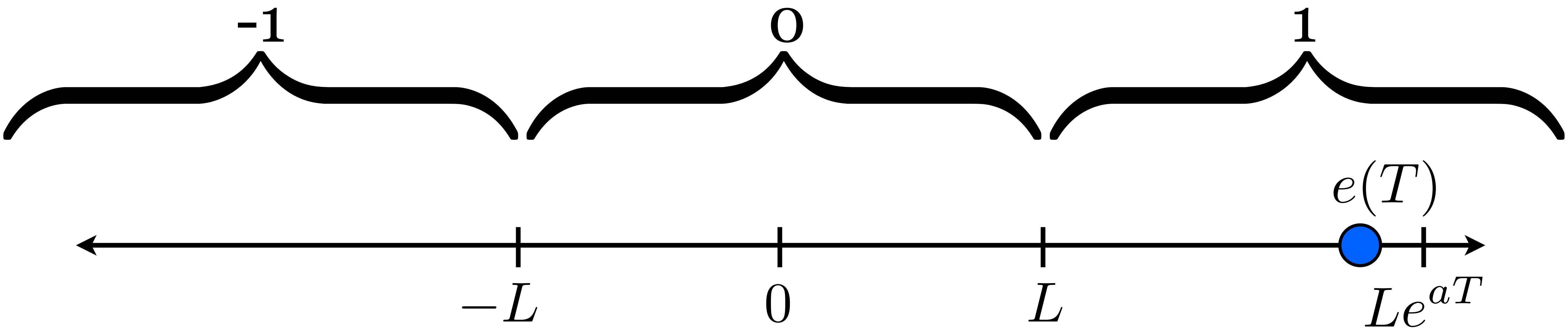

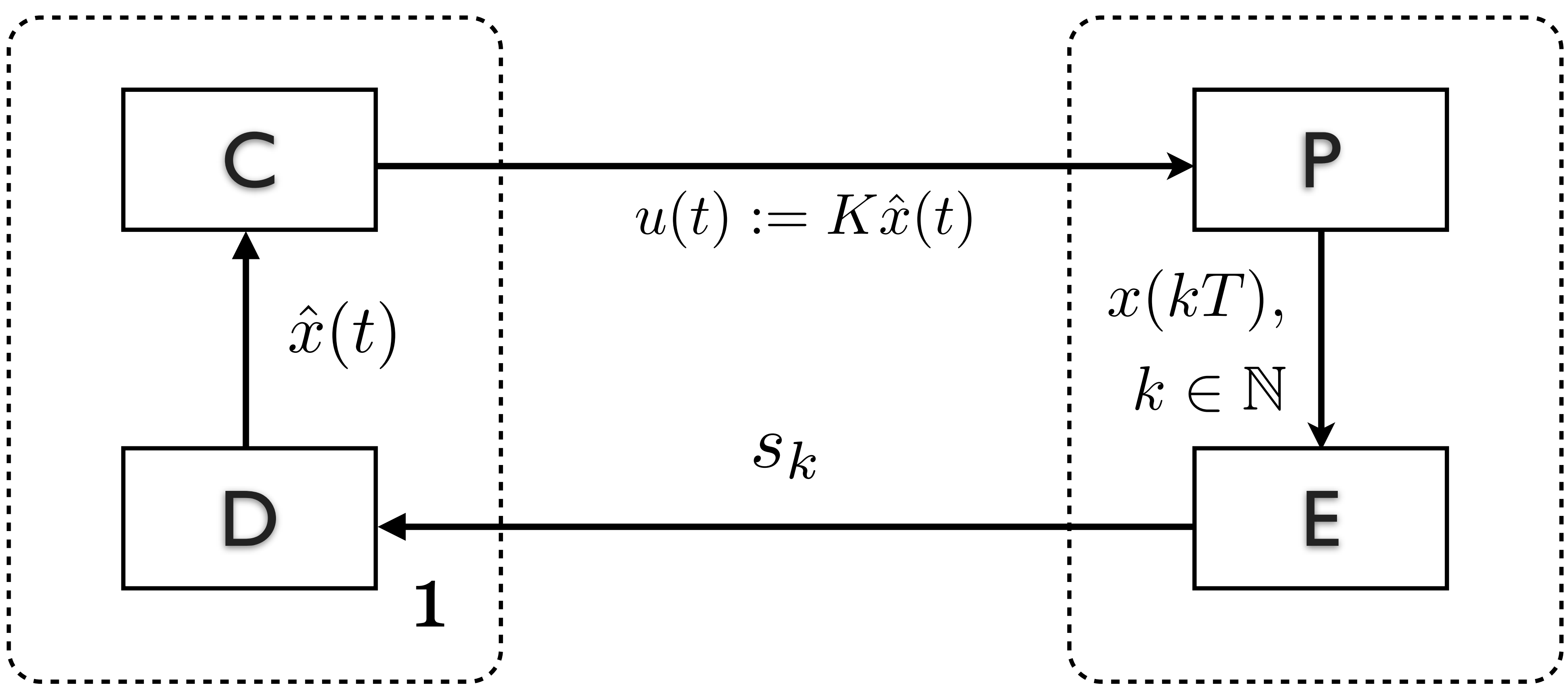

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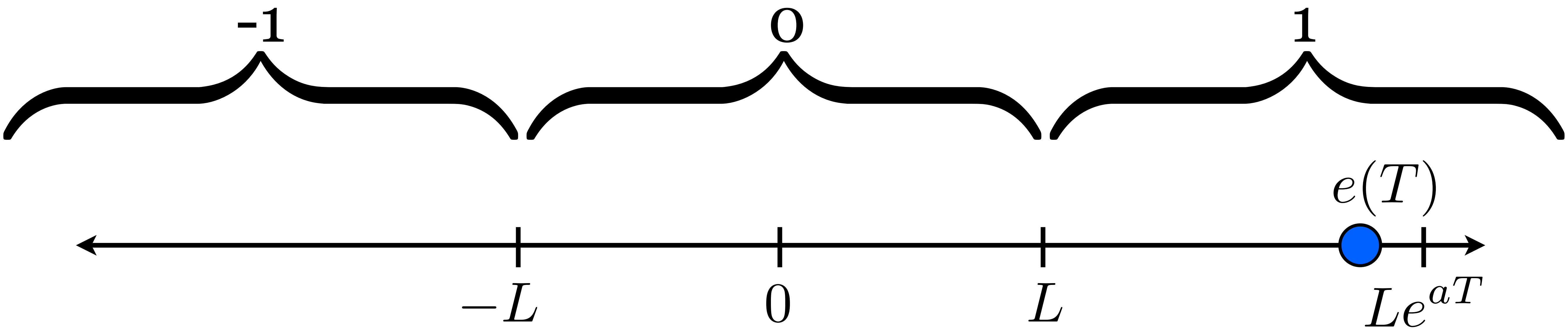

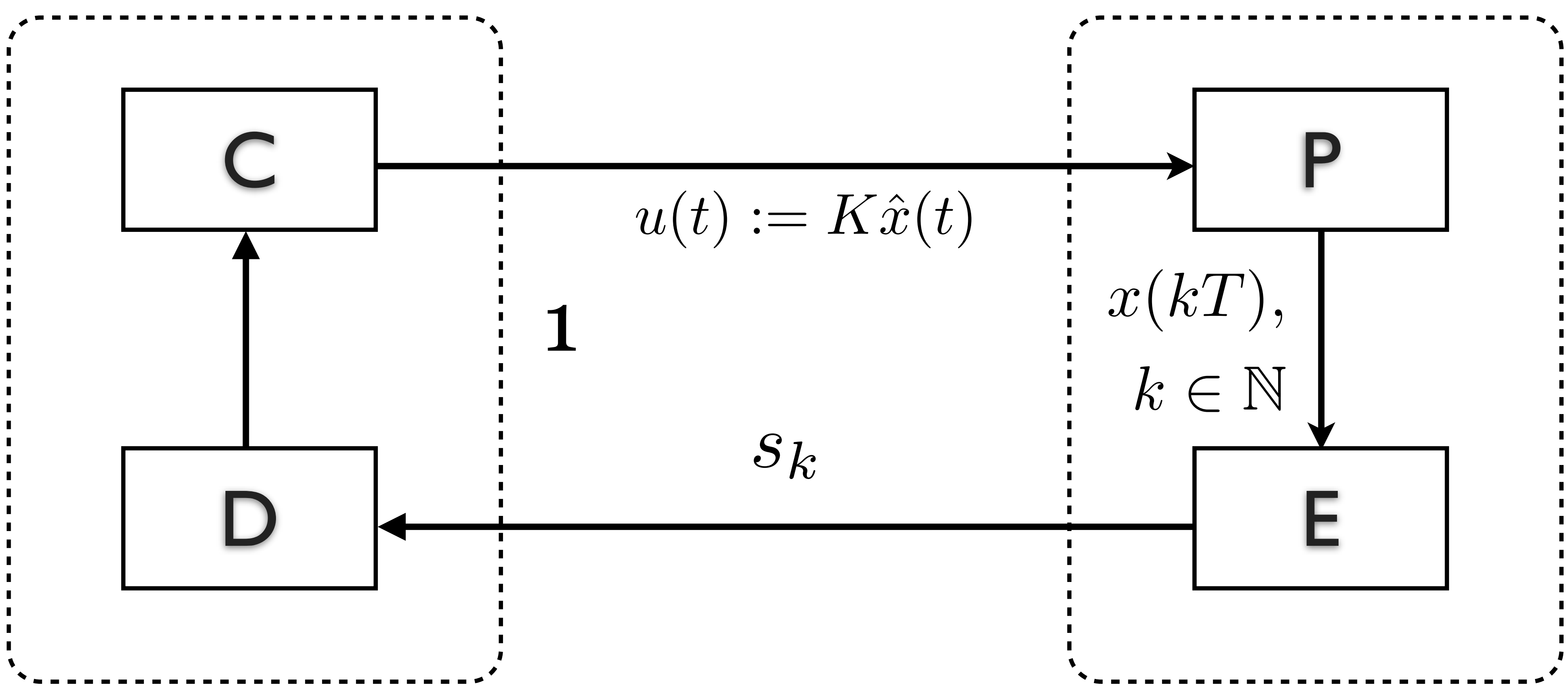

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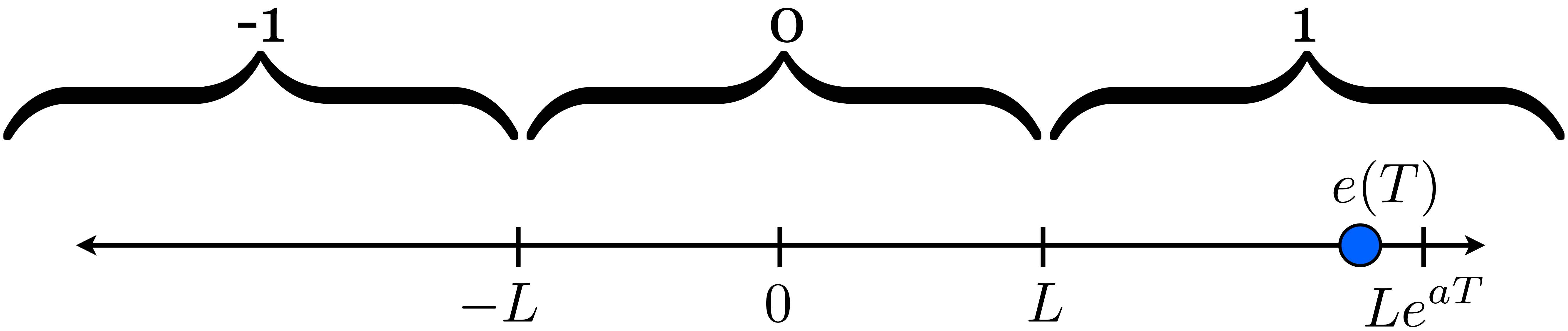

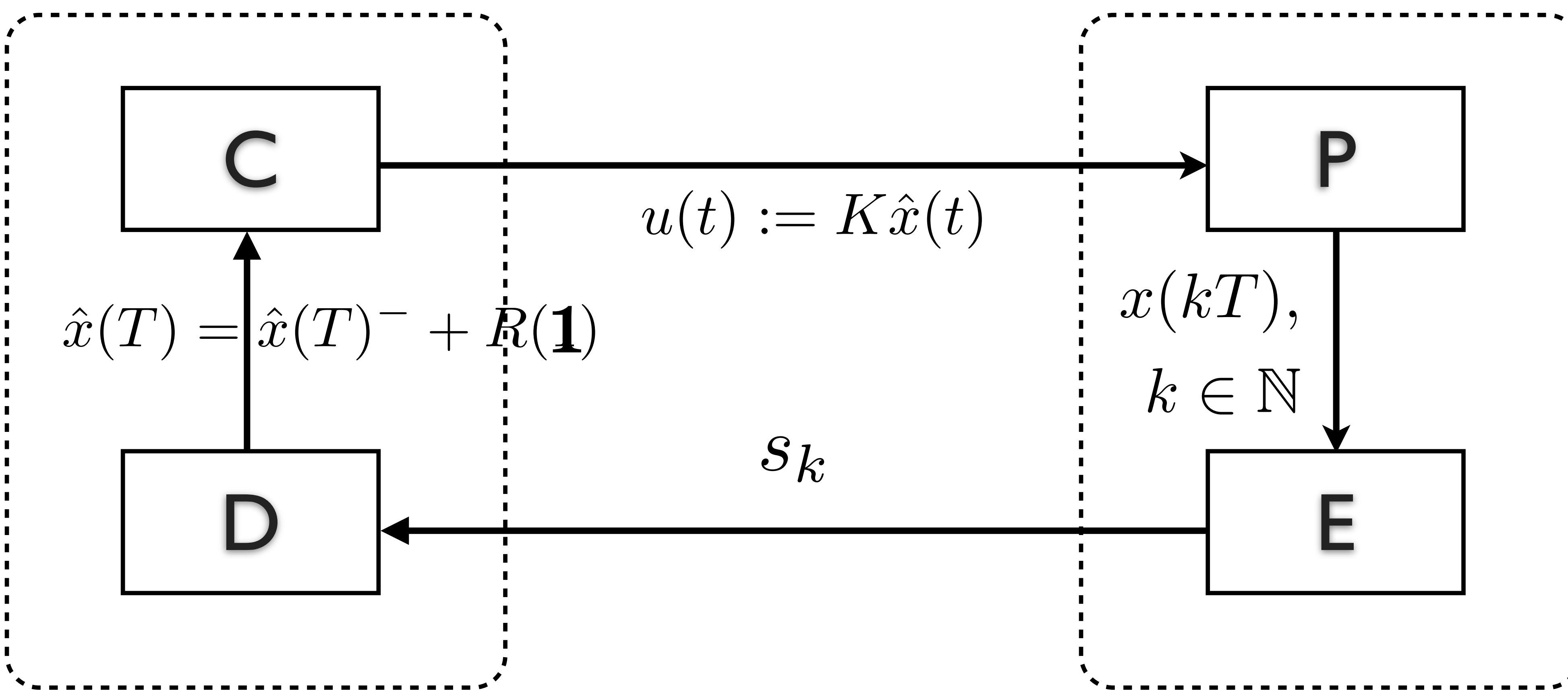

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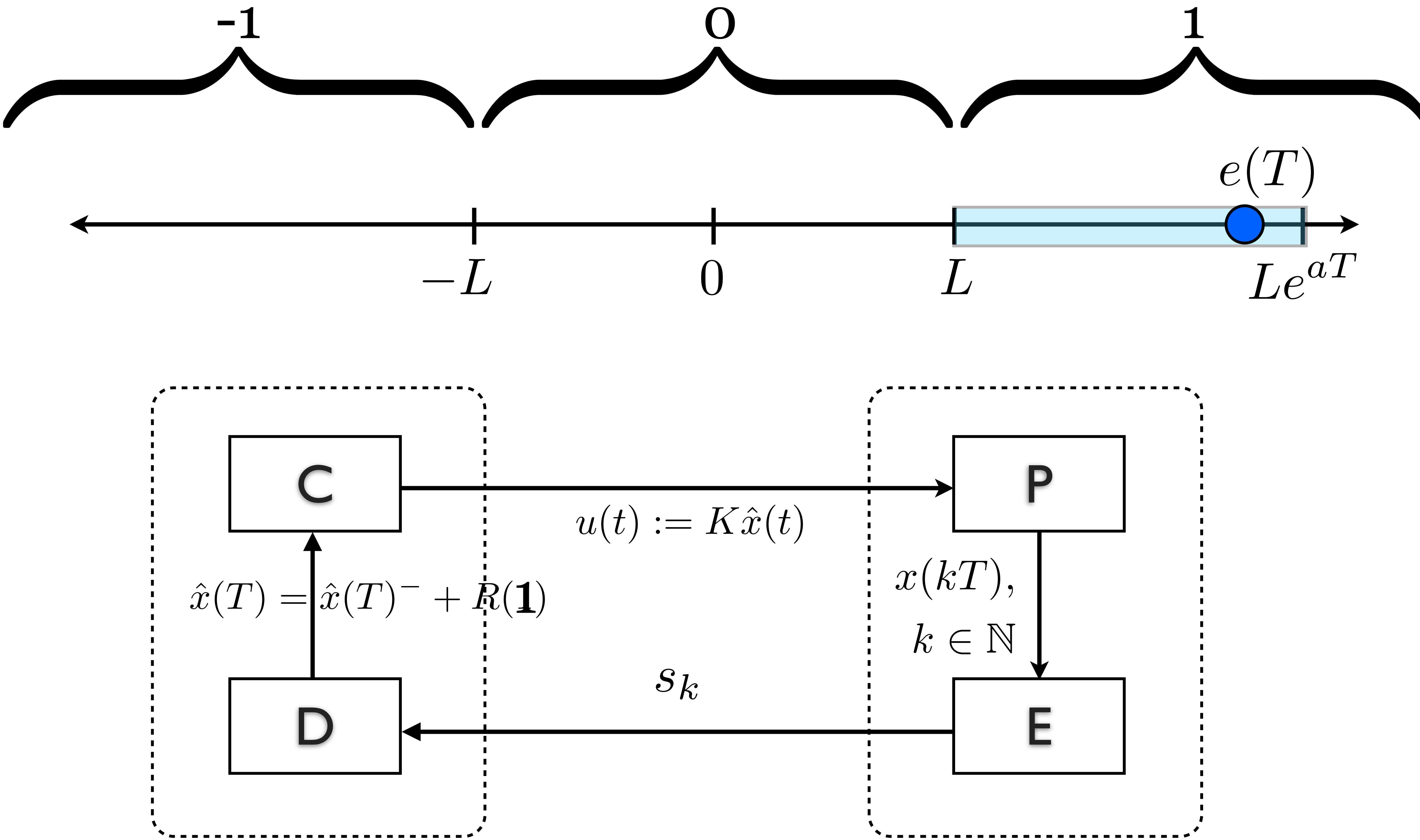

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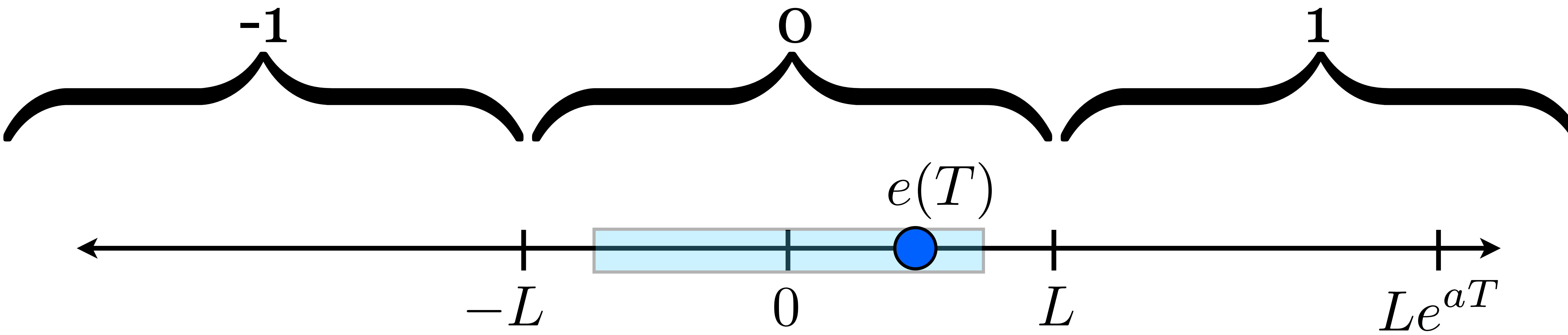
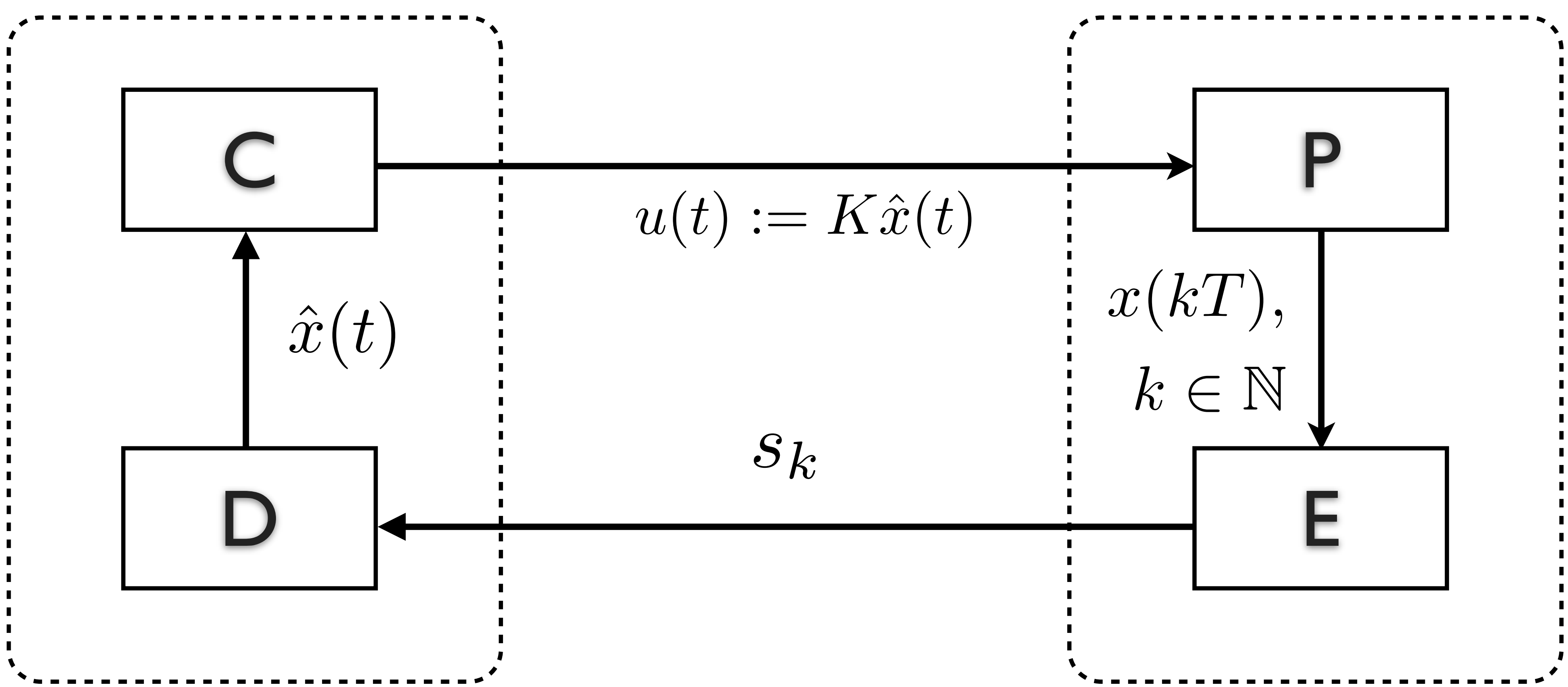
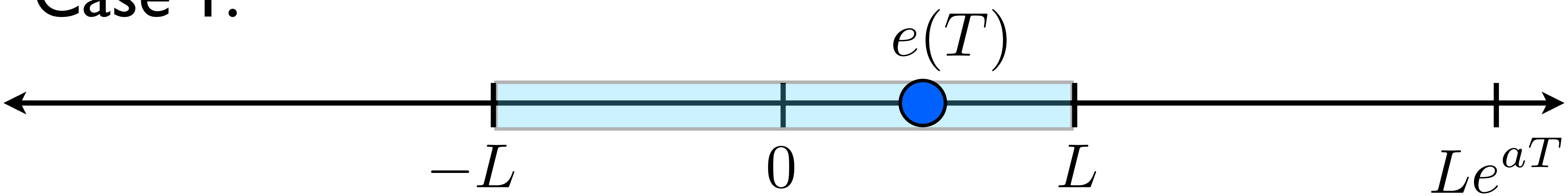




Event-based Results

Case 1:



If r and A satisfy

$$r \;\; \ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

then an emulation-based controller and event-based encoder/decoder pair exist with bit-rate r which bound x(t)

Event-based Results

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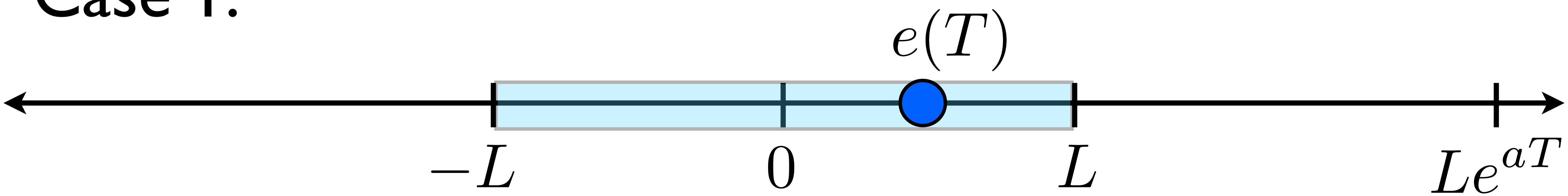

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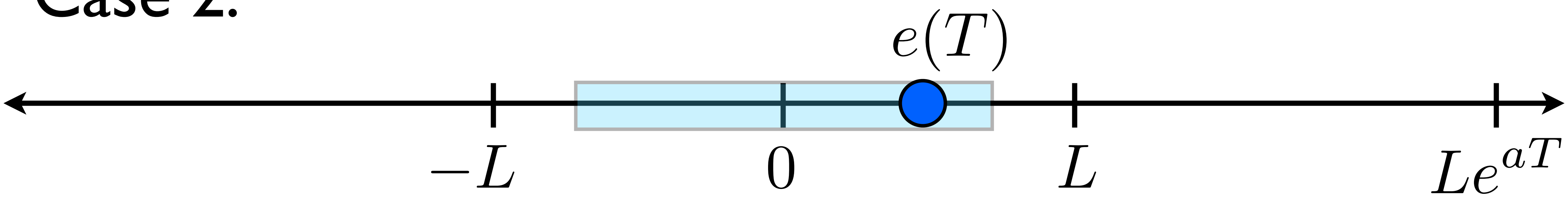
(encoder will have γ=1)

Event-based Results

Case 2:



If r, γ_{max}, and A satisfy

$$r \boxed{\frac{h^{-1}(\gamma_{\max})}{\ln 3}} \ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A] \qquad h(x) := \frac{1}{1+\frac{1}{x}\ln\frac{2}{e^x-1}}$$

new

then an emulation-based controller and event-based encoder/decoder pair exist with bit-rate r and with ave. comm. <= γ_{max} which bound x(t)

Event-based Results

Compare the two lower bounds on bit-rate penalty factor

$$r \, \frac{h^{-1}(\gamma_{\max})}{\ln 3} \, \ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

↕

event-based: $$\frac{r}{r_{\min}} \geq \frac{1}{\frac{h^{-1}(\gamma_{\max})}{\ln 3}}$$

nec/suff: $$\frac{r}{r_{\min}} \geq \frac{1}{f(\gamma_{\max}, S)}$$

Event-based Results

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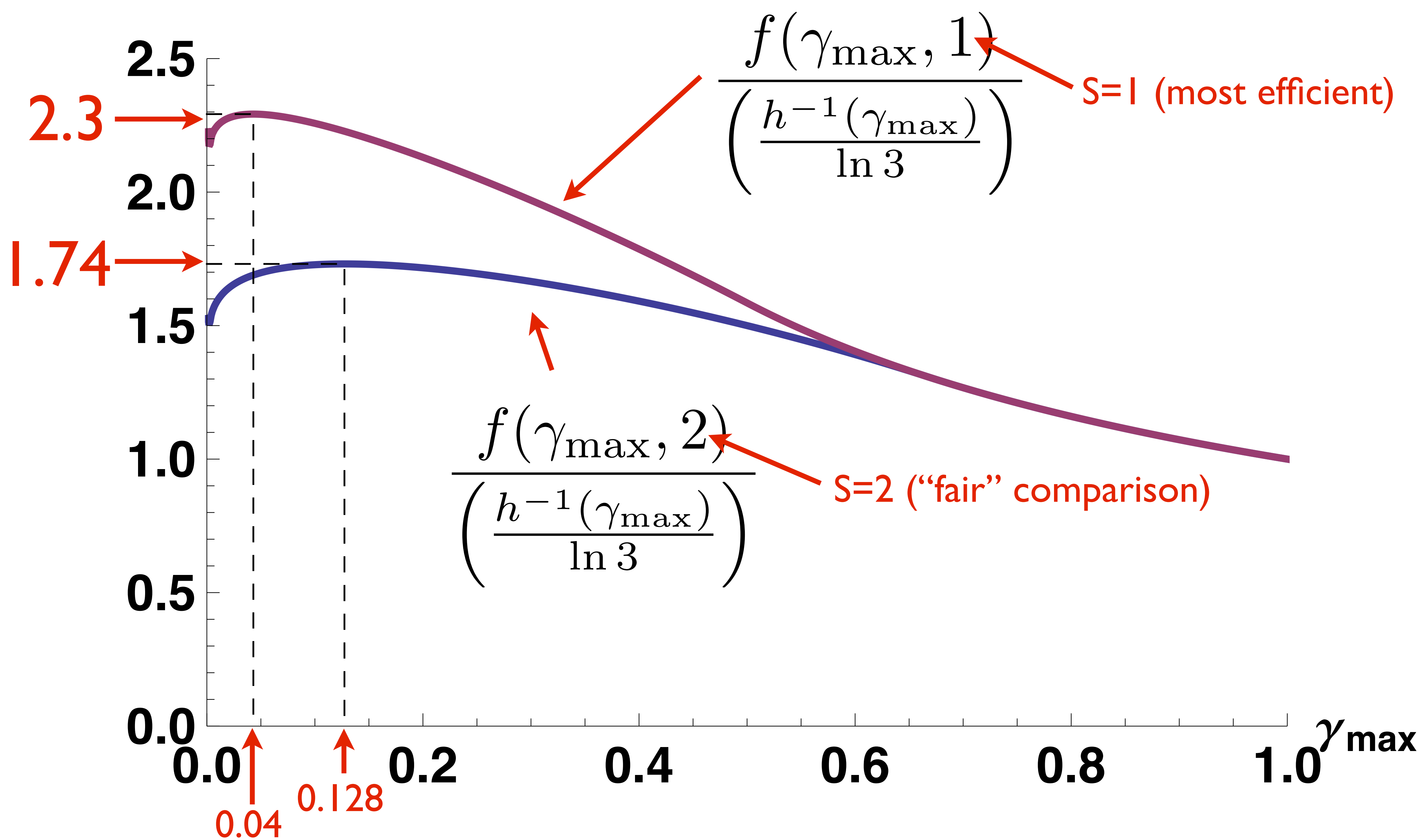
nec/suff: $$\frac{r}{r_{\min}} \geq \frac{1}{f(\gamma_{\max}, S)}$$

ratio: how conservative is event-based?

Event-based Results

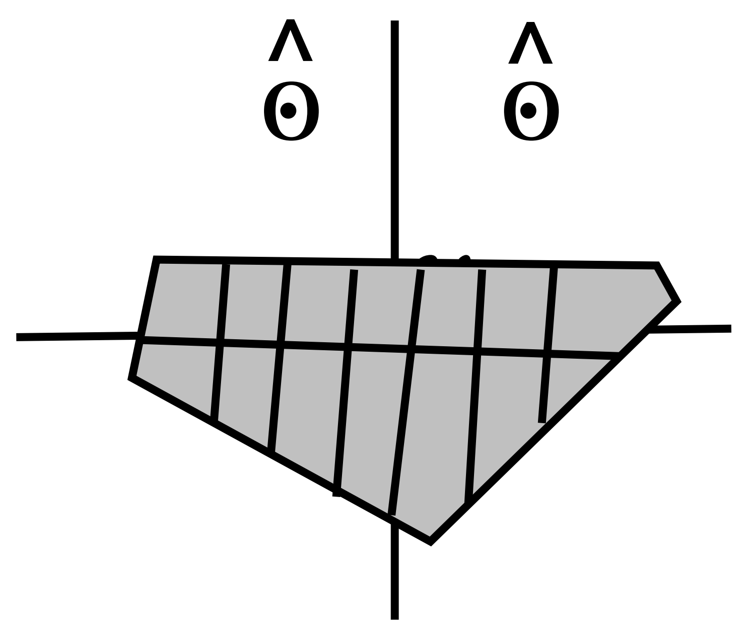
Event-based encoder has a bit-rate <2.3x that of any other bounding encoder with ave comm γ_{max}



Conclusion

- Control a linear system under bit-rate & communication constraints
- Nec/suff condition for a bounding enc/dec
- Easily-implemented event-based enc/dec within 2.3x of lowest possible bit-rate

Thanks



Backup/Cool Slides

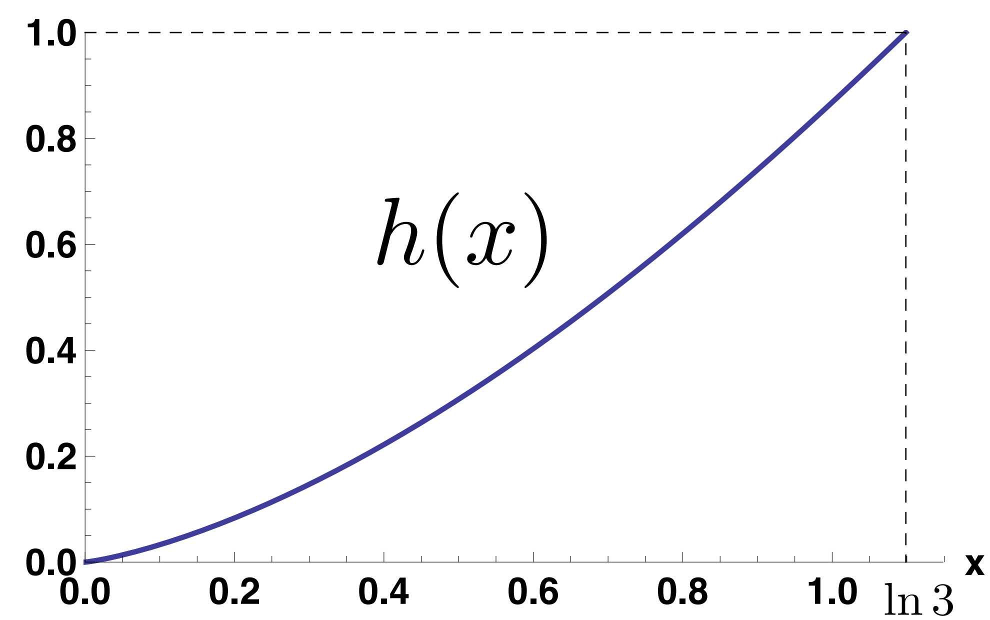
Event-based Results

If r, γ, and A satisfy

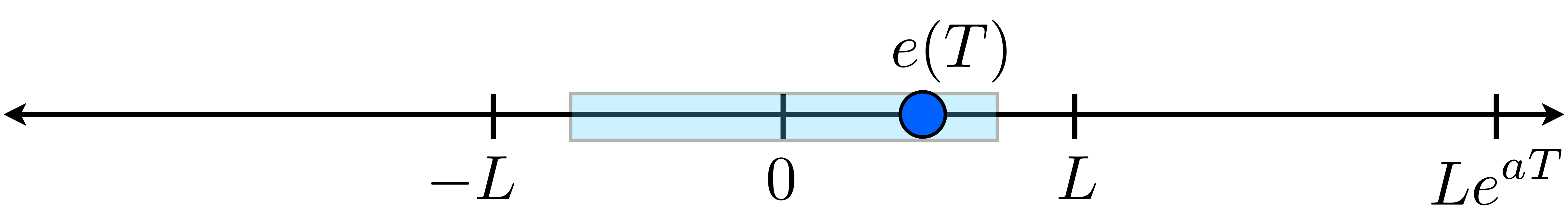
$$r\boxed{\frac{h^{-1}(\gamma_{\max})}{\ln 3}}\ln 2 \geq \sum_{i:\Re\lambda_i[A]\geq 0} \lambda_i[A]$$

new

$$h(x) := \frac{1}{1+\frac{1}{x}\ln\frac{2}{e^x-1}}$$



then an emulation-based controller and event-based encoder/decoder pair exist with bit-rate r and with ave. comm <= γ which bound x(t)



Example

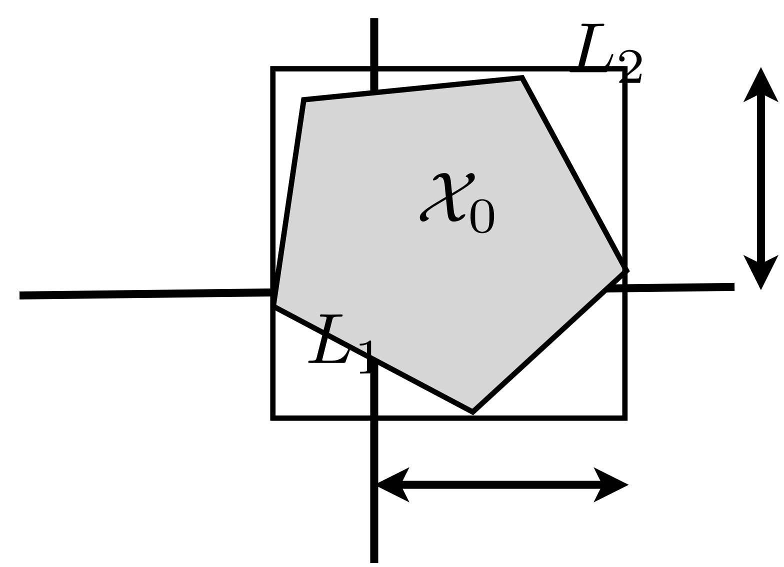
- S=1 (bits)
- (animated example?)
- N=5 (show codewords?)
- Show M-of-N encoders?
- show example of {0,1}, T=0.1 versus {0,...,64000}, T=1 which have the same bit-rate but the bigger alphabet is not better.

Event-based enc/dec

- The enc has n "sub-encoders" which send one of {-1,0,1} every Ti sec if |xi| > Li, with

$$T_i := \frac{h^{-1}(\gamma_{\max})}{\lambda_i[A]}$$

$$L_i := \sup_{z \in \mathcal{X}_0} |z_i|,$$



Event-based enc/dec

At each timestep $t_{i,k} \coloneqq T_i k$ with $i \in \{1,\ldots,n\}, k \in \mathbb{N}$, the ith sub-encoder sends symbol $s_{i,k} \in \{-1,0,1\}$ according to

$$s_{i,k} = \begin{cases} -1 & e_i(T_i k) < -L_i \\ 0 & e_i(T_i k) \in [-L_i, L_i] \\ 1 & e_i(T_i k) > L_i \end{cases}, \qquad i \in \{1,\ldots,n\}, \quad k \in \mathbb{N}. \tag{1}$$

This concludes the description of the encoder. Unlike the encoder, the decoder does not have access to $x(t)$, so it cannot compute the estimation error $e(t)$. It has access to only its own internal state estimate $\hat{x}(t)$, the received symbols $s_{i,k}$, and each sub-encoder's T_i and L_i. At timestep $t_{i,k} \coloneqq T_i k$, the decoder receives symbol $s_{i,k}$ and at that time it and the encoder each update the ith component of their state estimates $\hat{x}_i(T_i k)$ as

$$\hat{x}_i(T_i k) = \hat{x}_i(T_i k)^- + R_i(s_{i,k}), \qquad i \in \{1,\ldots n\},\ k \in \mathbb{N}, \tag{2}$$

where for each dimension i, the decoding function $R_i : \mathcal{A} \to \mathbb{R}$ is defined as

$$R_i(s) \coloneqq \begin{cases} -\frac{L_i}{2}(1+\exp(h^{-1}(\gamma_{\max}))) & s=-1 \\ 0 & s=0 \\ \frac{L_i}{2}(1+\exp(h^{-1}(\gamma_{\max}))) & s=1. \end{cases} \tag{3}$$

(N,M,S) Encoders

- Introduce (N,M,S) encoders
- Show that any S-library encoder is (N,M,S) for some N,M
- Show that L(N,N gam,S) --> H(X)